Mathematics 1214: Introduction to Group Theory

Homework exercise sheet 1

Due 12:50pm, Friday 29th January 2010

1. List all of the mappings $\alpha\colon \{1,2\} \to \{a,b\}$. Which of these are injective, which are surjective and which are bijective?

   Solution These are: $\alpha_1\colon 1 \mapsto a,\ 2 \mapsto b$, $\alpha_2\colon 1 \mapsto b,\ 2 \mapsto a$, $\alpha_3\colon 1 \mapsto a,\ 2 \mapsto a$ and $\alpha_4\colon 1 \mapsto b,\ 2 \mapsto b$.

   α_1 and α_2 are bijective, so they are injective and surjective.

   α_3 is not injective since $\alpha_3(1) = \alpha_3(2)$, and it is not surjective since $\alpha(x) \neq b$ for any $x \in \{1,2\}$.

   Similarly, α_4 is not injective or surjective.

2. Consider the mapping $\alpha\colon \mathbb{N} \to \mathbb{N}$, $n \mapsto n^2$ where $\mathbb{N} = \{1,2,3,\dots\}$ is the set of positive integers.

   (a) Is α injective? Is α surjective?

   (b) Construct a function $\beta\colon \mathbb{N} \to \mathbb{N}$ such that $\beta\circ\alpha = \iota_{\mathbb{N}}$, and check that $\alpha\circ\beta \neq \iota_{\mathbb{N}}$.

   Solution (a) If $n, m \in \mathbb{N}$ with $\alpha(n) = \alpha(m)$ then $n^2 = m^2$. Since $n, m \geq 0$, this implies that $n = m$. So α is injective.

   Clearly, $\alpha(n) \neq 2$ for any $n \in \mathbb{N}$. So α is not surjective.

   (b) Let $\beta\colon \mathbb{N} \to \mathbb{N}$ be given by

   $$\beta(m) = \begin{cases} n & \text{if } m = n^2 \text{ for some } n \in \mathbb{N}, \\ 1 & \text{if } m \neq n^2 \text{ for every } n \in \mathbb{N}. \end{cases}$$

   Note that for every $m \in \mathbb{N}$, we have $m = n^2$ for at most one $n \in \mathbb{N}$. So this formula defines a mapping $\beta\colon \mathbb{N} \to \mathbb{N}$.

   Then we have $\beta \circ \alpha\colon \mathbb{N} \to \mathbb{N}$, and for every $n \in \mathbb{N}$, we have $\beta \circ \alpha(n) = \beta(n^2) = n$. Since $\iota_{\mathbb{N}}\colon \mathbb{N} \to \mathbb{N}$ and $\iota_{\mathbb{N}}(n) = n$ for every $n \in \mathbb{N}$, this shows that the domains and codomains of $\beta \circ \alpha$ and ι_S are equal, and that they take the same values. So $\beta \circ \alpha = \iota_{\mathbb{N}}$.

   On the other hand, $\alpha \circ \beta(2) = \alpha(1) = 1 \neq 2 = \iota_{\mathbb{N}}(2)$, so $\alpha \circ \beta \neq \iota_{\mathbb{N}}$.

3. Give examples of mappings $\mathbb{R} \to \mathbb{R}$ which are

   (a) bijective;

   (b) injective but not surjective;

   (c) surjective but not injective;

   (d) neither injective nor surjective.

   Be sure to explain why your answers are correct.

Solution (a) For example, $\iota_{\mathbb{R}}$ is a bijection $\mathbb{R} \to \mathbb{R}$, since ι_S is a bijections $S \to S$ for any set S.

(b) For example, $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^x$ is injective, since if $x, y \in \mathbb{R}$ with $f(x) = f(y)$ then $e^x = e^y$, so $e^{x-y} = 0$, so $x - y = 0$, so $x = y$. The function f is not surjective, since $e^x > 0$ for all $x \in \mathbb{R}$, so (for example) there is no $x \in \mathbb{R}$ with $f(x) = -1$.

(c) For example,

$$g\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} \log(x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

is surjective since if $y \in \mathbb{R}$ then $x = e^y$ satisfies $g(x) = \log(e^y) = y$. It is not injective, since $g(-1) = g(0) = 0$.

(d) For example, $k\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto 0$ is not surjective, since $k(x) \neq 1$ for all $x \in \mathbb{R}$, and it is not injective since $k(0) = k(1)$.

4. Let $m \in \mathbb{N}$. How many surjective mappings are there from $\{1, 2, \ldots, m\}$ to $\{1, 2\}$?

   Solution Let $S = \{1, 2, \ldots, m\}$. There are 2^m mappings $S \to \{1, 2\}$, and if $\alpha\colon S \to \{1, 2\}$ is not surjective then either $\alpha(x) \neq 2$ for all $x \in S$, or $\alpha(x) \neq 1$ for all $x \in S$. In the first case, $\alpha(x) = 1$ for all $x \in S$, and in the second case, $\alpha(x) = 2$ for all $x \in S$. So the only non-surjective mappings $S \to \{1, 2\}$ are the two constant functions. Hence the number of surjective mappings is $2^m - 2$.

5. Let S, T be sets with $S \neq \emptyset$ and let $\alpha\colon S \to T$. Prove that α is one-to-one if and only if there is a function $\beta\colon T \to S$ such that $\beta \circ \alpha = \iota_S$.

   Solution Suppose that α is one-to-one. Since S is non-empty, we can fix some $x_0 \in S$. If $y \in T$ then since α is one-to-one, there is at most one $x \in S$ such that $\alpha(x) = y$. If there is exactly one $x \in S$ such that $\alpha(x) = y$, let us define $\beta(y) = x$, and if there is no $x \in S$ such that $\alpha(x) = y$, let us define $\beta(y) = x_0$.

   We claim that β is then a mapping $T \to S$ and $\beta \circ \alpha = \iota_A$. Indeed, for every $y \in T$ we have given precisely one value $x \in S$ associated by β to y, so β is a mapping $T \to S$, and if $x \in S$ then $\beta \circ \alpha(x) = \beta(\alpha(x)) = x$ by the definition of β. So $\beta \circ \alpha = \iota_S$.

6. An operation $*$ on a set S is said to be *commutative* if $a * b = b * a$ for all $a, b \in S$.

   Complete the following Cayley table so that the operation $\bullet$ on $\{1, 5, 6\}$ is commutative:

$\bullet$	1	5	6
1		1	6
5		6	
6		5	

How many different ways are there of doing this?

Solution Here is one completion:

$\bullet$	1	5	6
1	1	1	6
5	1	6	5
6	6	5	1

The red numbers are forced by the condition that $a \bullet b = b \bullet a$. For example, the (1,5)-entry of the table tells us that $1 \bullet 5 = 1$, so we need $5 \bullet 1 = 1$ which forces the (5,1)-entry to be 1.

On the other hand, the blue entries can be any elements of $\{1, 5, 6\}$. Since this set has three elements, the number of different completions yielding a commutative operation is $3^2 = 9$.

7. Let $M(2, \mathbb{R})$ be the set of 2×2 matrices with real entries, and let $*$ be the operation on $M(2, \mathbb{R})$ defined by $A * B = AB - BA$ for $A, B \in M(2, \mathbb{R})$.

   (a) Find three matrices $A, B, C \in M(2, \mathbb{R})$ such that $A * (B * C) = (A * B) * C$.

   (b) Find three matrices $A, B, C \in M(2, \mathbb{R})$ such that $A * (B * C) \neq (A * B) * C$.

Solution (a) For example, let $A = B = C = 0$, by which we mean the zero matrix $0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since $0 * 0 = 0$, this gives $A * (B * C) = 0 = (A * B) * C$.

(b) For example, let $A = B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and let $C = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We have $A * B = AB - BA = A^2 - A^2 = 0$, so $(A * A) * C = 0 * C = 0$, but $A * C = B * C = C$, so $A * (B * C) = A * C = C \neq 0$.