**Mathematics 1214: Introduction to Group Theory**

Solutions to exercise sheet 11

1. Let $n \geq 3$ and let $D_n$ be the dihedral group of order $2n$. Writing $\rho = \rho_{2\pi/n}$, show that $\langle\rho\rangle$ is a normal subgroup of $D_n$. [Hint: if $r$ is reflection in the $x$-axis, then $r \circ \rho = \rho^{-1} \circ r$, and the reflections in $D_n$ are $r_j = \rho^j \circ r$ for $0 \leq j < n$.] What is the order of the quotient group $D_n/\langle\rho\rangle$?

   **Solution** Let $\theta\colon D_n \to \{-1, 1\}$ be given by defining $\theta(\alpha) = 1$ if $\alpha$ is a proper motion, and $\theta(\alpha) = -1$ if $\alpha$ is an improper motion. By homework exercise sheet 5, question 5(d), $\theta$ is then a homomorphism into the group $\{-1, 1\}$ with multiplication. Moreover, the proper motions in $D_n$ are precisely $e, \rho, \rho^2, \dots, \rho^{n-1}$ (since these are orientation-preserving elements of $D_n$, and the remaining elements are all reflections, hence orientation-reversing), so $\ker\theta = \{\alpha \in D_n : \theta(\alpha) = 1\} = \langle\rho\rangle$, hence $\langle\rho\rangle$ is a normal subgroup of $D_n$ by Theorem 47.

   Alternatively, we could avoid having to dream up a suitable homomorphism, and instead use the definition of "normal subgroup" directly. Let $N = \langle\rho\rangle$. We have to check that if $n \in N$ and $g \in D_n$ then $gng^{-1} \in N$.

   If $g \in N$ then $gng^{-1} \in N$, since $N$ is a subgroup.

   If $g \notin N$ then $g$ is one of the reflections in $D_n$, say $g = r_j = \rho^j \circ r$. Then $g^{-1} = (\rho^j \circ r)^{-1} = r^{-1} \circ (\rho^j)^{-1} = r \circ \rho^{-j}$. Moreover, $n \in N = \langle\rho\rangle$, so $n = \rho^k$ for some $k \in \mathbb{Z}$. So $gng^{-1} = \rho^j \circ r \circ \rho^k \circ r \circ \rho^{-j}$. Since $r \circ \rho = \rho^{-1} \circ r$, we have $r \circ \rho^2 = r \circ \rho \circ \rho = \rho^{-1} \circ r \circ \rho = \rho^{-1} \circ \rho^{-1} \circ r = \rho^{-2} \circ r$, and similarly, we have $r \circ \rho^k = \rho^{-k} \circ r$. Hence $gng^{-1} = \rho^j \circ \rho^{-k} \circ r \circ r \circ \rho^{-j} = \rho^j \circ \rho^{-k} \circ \rho^{-j} = \rho^{j-k-j} = \rho^{-k}$, which is in $N = \langle\rho\rangle$. So $gng^{-1} \in N$.

   Hence $N$ is normal in $D_n$.

   Finally, we have $|D_n| = 2n$ and $|\langle\rho\rangle| = o(\rho) = n$, so $|D_n/\langle\rho\rangle| = \frac{|D_n|}{|\langle\rho\rangle|} = \frac{2n}{n} = 2$.

2. Give an example of a normal subgroup of $GL(2, \mathbb{R})$, and give an example of a subgroup of $GL(2, \mathbb{R})$ which is not normal.

   **Solution** Trivial examples of normal subgroups are $GL(2, \mathbb{R})$ itself, and $\{e_{GL(2,\mathbb{R})}\} = \{I\}$. A less trivial example is $SL(2, \mathbb{R})$, which is the kernel of the determinant and so a normal subgroup (since the determinant is a homomorphism).

   Consider $H = \{\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] : x \in \mathbb{R}\}$. This is a subgroup of $GL(2, \mathbb{R})$ [you should check this] but taking $X = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in H$ and $A = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \in GL(2, \mathbb{R})$ gives $A^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$ and $AXA^{-1} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right] \notin H$. So $H$ is not normal in $GL(2, \mathbb{R})$.

3. Let $G$ be a group and let $N$ be a normal subgroup of $G$.

   (a) Prove that the mapping $\eta\colon G \to G/N$ defined by $\eta(a) = Na$ for $a \in G$ is a homomorphism. What are the kernel and image of $\eta$?

   (b) If $H$ is another group and $\theta\colon G \to H$ is a homomorphism with kernel $N$, and $\varphi\colon G/N \to H$ is defined by $\varphi(Na) = \theta(a)$, then $\varphi$ is a well-defined homomorphism (see the proof of the Fundamental Homomorphism Theorem). Prove that $\theta = \varphi \circ \eta$.

**Solution** (a) For all $a, b \in G$ we have $\eta(ab) = (Na)(Nb) = N(ab) = \eta(ab)$, so $\eta$ is a homomorphism. The image of $\eta$ is $\eta(G) = \{\eta(a) \colon a \in G\} = \{Na \colon a \in G\} = G/N$ and the kernel is $\ker \eta = \{a \in G \colon Na = e_{G/N} = N\} = \{a \in G \colon a \in N\} = N$ since $Na = N \iff a \in N$ (if you like, by Theorem 35 applied with $H = N$ and $b = e$).

(b) For any $a \in G$ we have $(\varphi \circ \eta)(a) = \varphi(\eta(a)) = \varphi(Na) = \theta(a)$ by the definitions of the mappings $\eta$ and $\varphi$. Since $\varphi \circ \eta$ has the same domain and codomain as $\theta$ and we've just seen that they take the same values at each point of their domains, this shows that $\varphi \circ \eta = \theta$.

4. Let $\theta \colon \mathbb{R} \to \mathbb{C}^\times$, $\theta(x) = e^{ix}$. Prove that $\theta$ is a homomorphism with kernel $\langle 2\pi \rangle$, and that $\mathbb{R}/\langle 2\pi \rangle \approx \{z \in \mathbb{C}^\times \colon |z| = 1\}$.

**Solution** For any $x, y \in \mathbb{R}$ we have $\theta(x+y) = e^{i(x+y)} = e^{ix}e^{iy} = \theta(x)\cdot\theta(y)$. So $\theta$ is a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C}^\times, \cdot)$. Its kernel is

$$\ker \theta = \{x \in \mathbb{R} \colon \theta(x) = e_{\mathbb{C}^\times} = 1\} = \{x \in \mathbb{R} \colon e^{ix} = 1\} = \{n \cdot 2\pi \colon n \in \mathbb{Z}\} = \langle 2\pi \rangle.$$

By the fundamental homomorphism theorem (actually, this is Corollary 50), we have

$$\mathbb{R}/\langle 2\pi \rangle \approx \theta(\mathbb{R}) = \{e^{ix} \colon x \in \mathbb{R}\} = \{z \in \mathbb{C}^\times \colon |z| = 1\}.$$

5. Let $G$ be a group with identity element $e$.

   (a) Show that $\{e\} \lhd G$ and $G/\{e\} \approx G$. (b) Show that $G \lhd G$ and $G/G \approx \{e\}$.

**Solution** (a) Let $\theta \colon G \to G$ be given by $\theta(a) = a$ for all $a \in G$ (so that $\theta$ is just the identity mapping on $G$). Then $\theta$ is a surjective homomorphism with kernel $\{e\}$, so $\{e\} \lhd G$ by Theorem 47, and $G/\{e\} \approx G$ by the fundamental homomorphism theorem.

(b) Let $\theta \colon G \to \{e\}$ be given by $\theta(a) = e$ for all $a \in G$. Then $\theta$ is a surjective homomorphism with kernel $G$, so $G \lhd G$ by Theorem 47 and $G/G \approx \{e\}$ by the fundamental homomorphism theorem.

6. Let $G$ be a group.

   (a) Show that if $N$ is a normal subgroup of $G$, then $G/N$ is an abelian group if and only if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

   (b) Give an example of a non-abelian group $G$ and a normal subgroup $N$ of $G$ such that $G/N$ is a finite abelian group. [You can always do this easily with $N = G$. So if you want a little more of a challenge, try to do it with $N \neq G$.]

   (c) Give another example of a non-abelian group $G$ and a normal subgroup $N$ of $G$ such that $G/N$ is an infinite abelian group.

**Solution** (a) If $a, b \in G$ then we have

$$(Na)(Nb) = (Nb)(Na) \iff N(ab) = N(ba) \iff (ab)(ba)^{-1} \in N \iff aba^{-1}b^{-1} \in N$$

where we have used Theorem 35 in the middle. So $G/N$ is abelian $\iff$ $(Na)(Nb) = (Nb)(Na)$ for all $a, b \in G$ $\iff$ $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

(b) For a trivial example, take $N = G = S_3$; then $G$ is non-abelian and $G/N = G/G \approx \{e\}$ and $\{e\}$ is finite and abelian, so $G$ is finite and abelian.

For a less trivial example, take $G = S_3 \times \mathbb{Z}_2$ and let $N = S_3 \times \{[0]\}$. Then $G$ is non-abelian since $S_3$ is non-abelian. Moreover, $N$ is a normal subgroup of $G$ and $G/N \approx \mathbb{Z}_2$. [You should justify the three statements in the last sentence by considering the map $\pi\colon S_3 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, $\pi((\alpha, [k])) = [k]$ for $(\alpha, [k]) \in S_3 \times \mathbb{Z}_2$]. Since $\mathbb{Z}_2$ is a finite and abelian, this shows that $G/N$ is finite and abelian.

(c) Let $G = S_3 \times \mathbb{Z}$ and let $N = S_3 \times \{[0]\}$. Then (much as in (b)) $G$ is non-abelian, $N \lhd G$ and $G/N \approx \mathbb{Z}$ is an infinite abelian group.

7. If $G$ is a cyclic group and $N$ is a subgroup of $G$, explain why $N$ is a normal subgroup of $G$ and prove that $G/N$ is a cyclic group. [Hint: for the second part, if $G = \langle a \rangle$, prove that $G/N = \langle Na \rangle$.]

**Solution** If $G$ is cyclic then $G$ is abelian. Any subgroup of an abelian group is normal. So any subgroup $N$ of $G$ is normal.

Since $G$ is cyclic, we have $G = \langle a \rangle = \{a^k \colon k \in \mathbb{Z}\}$ for some $a \in G$. So

$$G/N = \{Nx \colon x \in G\} = \{N(a^k) \colon k \in \mathbb{Z}\} = \{(Na)^k \colon k \in \mathbb{Z}\} = \langle Na \rangle.$$

(if you like, you can justify the third equality by applying Exercise 3(a) with Theorem 40(c).)

Hence $G/N$ is cyclic.

8. Let $G$ be a group and let $N \lhd G$.

   (a) If $G$ is a finite group, prove that $G/N$ is a finite group.

   (b) If $G$ is an infinite group and $N$ is a finite normal subgroup, prove that $G/N$ is an infinite group.

   (c) Show (by giving examples) that if $G$ is an infinite group and $N$ is an infinite normal subgroup, then $G/N$ may be finite, or it may be infinite.

**Solution** (a) $G/N$ is a partition of $G$, so if $G$ is finite then so is $G/N$.

Alternatively, if $n = |G|$ then we may write $G = \{a_1, \ldots, a_n\}$ and so $G/N = \{Na \colon a \in G\} = \{Na_1, \ldots, Na_n\}$ is a set of size at most $n$. [Note that $G/N$ will usually have size much smaller than $n$, since for many values of $i, j$ we will have $Na_i = Na_j$].

Alternatively, simply recall that if $G$ is finite then $|G/N| = \frac{|G|}{|N|}$. Since $|N| \geq 1$, we have $|G/N| \leq |G|$, so $G/N$ is a finite group.

(b) Again, $G/N$ is a partition of $G$. Since $G$ is an infinite set and each coset $Na$ in $G/N$ has the same size as $N$, which is finite, there must be infinitely many sets in this partition. So $G/N$ is infinite.

(c) If $G$ is any infinite group, then $G/G$ is finite (in fact, it has order 1).

If $G = \mathbb{Z} \times \mathbb{Z}$ and $N = \mathbb{Z} \times \{0\}$ then $N \lhd G$ and $G/N \approx \mathbb{Z}$ is an infinite group.