Mathematics 1214: Introduction to Group Theory

Solutions to homework exercise sheet 10

1. (a) Determine, with justification (and without using the Fundamental Theorem of Abelian Groups), which of the following groups are isomorphic, and which are not isomorphic. [Hint: think about the orders of elements of these groups].

$$\mathbb{Z}_8, \quad \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(b) Explain how the Fundamental Theorem of Abelian Groups can be used to answer (a) immediately.

Solution (a) Every element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has order 1 or 2. For example, $([1], [0], [1])^2 = ([1] \oplus [1], [0] \oplus [0], [1] \oplus [1]) = ([0], [0], [0])$ is the identity element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, so $([1], [0], [1])$ has order 2.

There's an element of $\mathbb{Z}_2 \times \mathbb{Z}_4$ of order 4, namely $([0]_2, [1]_4)$, but there's no element of order 8 in $\mathbb{Z}_2 \times \mathbb{Z}_4$.

The element $[1]_8$ of $\mathbb{Z}_8$ has order 8.

Hence the three groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_8$ are not isomorphic, by Theorem 41(d).

On the other hand, it's easy to show that $G \times H$ is always isomorphic to $H \times G$, since the mapping $(g, h) \mapsto (h, g)$ is an isomorphism. [Check this!] So $\mathbb{Z}_2 \times \mathbb{Z}_4 \approx \mathbb{Z}_4 \times \mathbb{Z}_2$. And $\mathbb{Z}_4 \times \mathbb{Z}_2$ is not isomorphic to any other group in the list, since if it were then $\mathbb{Z}_2 \times \mathbb{Z}_4$ would be too (by transitivity of $\approx$), but we've already shown that this is not true.

(b) The prime factorisation of 8 is $8 = 2^3$, so by the FTAG, every abelian group of order 8 is isomorphic to $\mathbb{Z}_{2^3}$ or $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and these groups aren't isomorphic. On the other hand, if we write $(n_1, n_2) = (2, 4)$ and $(m_1, m_2) = (4, 2)$, then (n_1, n_2) is a permutation of (m_1, m_2), so $\mathbb{Z}_2 \times \mathbb{Z}_4$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.

2. Use the Fundamental Theorem of Abelian Groups to list the abelian groups of order 37926 up to isomorphism. In other words, write down a list of abelian groups of order 37926 such that (1) no two groups in your list are isomorphic, but (2) every abelian group of order 37926 is isomorphic to one of the groups in your list. Hint: $37926 = 2 \cdot 3^2 \cdot 7^2 \cdot 43$.

Solution By the FTAG, the groups are:

$$\mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_{49} \times \mathbb{Z}_{43}, \quad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{49} \times \mathbb{Z}_{43}, \quad \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{43}, \quad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{43}.$$

3. Which of the following mappings are homomorphisms? For each homomorphism, compute its kernel and its image, and determine if it is injective and/or surjective, and whether or not it is an isomorphism.

(a) $\theta\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ [where, as usual, $\mathbb{R} = (\mathbb{R}, +)$]

(b) $\theta\colon \mathbb{R}^\times \to \mathbb{R}^\times$, $x \mapsto |x|$ [where, as usual, $\mathbb{R}^\times = (\mathbb{R}^\times, \cdot)$ is the group of non-zero real numbers under multiplication]

(c) $\theta\colon \mathbb{Z}_8 \to \mathbb{Z}_2 \times \mathbb{Z}_4$, $[k]_8 \mapsto ([k]_2, [k]_4)$ for $0 \le k < 8$

(d) $\theta\colon \mathbb{Z} \to SL(2,\mathbb{R})$, $n \mapsto \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$

(e) $\exp\colon \mathbb{R} \to (0,\infty)$, $x \mapsto \exp(x)$
[where $(0,\infty)$ is the group of positive real numbers under multiplication, and $\exp(x)$ is the exponential of x, sometimes written as e^x].

(f) $\theta\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x,y) \mapsto x - y$

Solution (a) This is not a homomorphism, since, for example, $\theta(-1+1) = \theta(0) = 0$ but $\theta(-1) + \theta(1) = 2$.

(b) We have $\theta(xy) = |xy| = |x| \cdot |y| = \theta(x) \cdot \theta(y)$. So θ is a homomorphism. We have $\ker\theta = \{x \in \mathbb{R}^\times \colon |x| = e_{\mathbb{R}^\times} = 1\} = \{1\}$, so θ is injective by Theorem 45. The image of θ is $\{|x| \colon x \in \mathbb{R}^\times\} = (0,\infty)$, so θ is not surjective. Hence it's not an isomorphism.

(c) We have $\theta([k]_8 \oplus [\ell]_8) = \theta([k+\ell]_8) = ([k+\ell]_2, [k+\ell]_4) = ([k]_2 \oplus [\ell]_2, [k]_4 \oplus [\ell]_4) = ([k]_2, [k]_4)([\ell]_2, [\ell]_4) = \theta([k]_8)\theta([\ell]_8)$, so θ is a homomorphism.
We have $\ker\theta = \{[k]_8 \colon ([k]_2, [k]_4) = e_{\mathbb{Z}_2 \times \mathbb{Z}_4} = ([0]_2, [0]_4)\} = \{[0]_8, [4]_8\}$, and the image of θ is $\theta(\mathbb{Z}_8) = \{([0]_2, [0]_8), ([1]_2, [1]_8), ([0]_2, [2]_8), ([1]_2, [3]_8)\}$ by direct computation. So θ is neither injective nor surjective. So it's certainly not an isomorphism.

(d) We have $\theta(n)\theta(m) = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & n+m \\ 0 & 1 \end{smallmatrix}\right) = \theta(n+m)$, so θ is a homomorphism (and note that $\det \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) = 1$, so θ is a well-defined mapping with codomain $SL(2,\mathbb{R})$). We have $\ker\theta = \{n \in \mathbb{Z} \colon \theta(n) = e_{SL(2,\mathbb{R})} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\} = \{0\}$, so θ is injective, and the image of θ is $\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \colon n \in \mathbb{Z}\}$ which is clearly not all of $SL(2,\mathbb{R})$, so θ is not surjective. So θ certainly isn't an isomorphism.

(e) We have $\exp(x+y) = e^{x+y} = e^x \cdot e^y = \exp(x)\exp(y)$, so exp is a homomorphism. We have $\ker\exp = \{x \in \mathbb{R} \colon \exp(x) = e_{(0,\infty)} = 1\} = \{0\}$, so exp is injective, and the image of exp is $\exp(\mathbb{R}) = \{e^x \colon x \in \mathbb{R}\} = (0,\infty)$. Hence exp is an isomorphism.

(f) We have $\theta((x,y)(x',y')) = \theta((x+x', y+y')) = x + x' - (y+y') = x - y + x' - y = \theta((x,y)) + \theta((x',y'))$. So θ is a homomorphism. Its kernel is $\ker\theta = \{(x,y) \colon \theta((x,y)) = e_{\mathbb{R}} = 0\} = \{(x,y) \colon x - y = 0\} = \{(x,x) \colon x \in \mathbb{R}\}$ (which is the graph of the function $y = x$) so θ is not injective, and its image is $\theta(\mathbb{R}) = \{x - y \colon x, y \in \mathbb{R}\} = \mathbb{R}$, so θ is surjective. It's not an isomorphism (since it's not injective).

4. Let G and H be two groups, let $\theta\colon G \to H$ be a homomorphism and consider the group $\theta(G)$.

(a) Prove that if G is a cyclic group, then so is $\theta(G)$.

(b) Disprove the statement: "if $n \in \mathbb{N}$ and G contains an element of order n, then so does $\theta(G)$" by finding a counterexample.

Solution (a) Since G is cyclic, $G = \langle a \rangle = \{a^k \colon k \in \mathbb{Z}\}$ for some $a \in G$. So $\theta(G) = \{\theta(g) \colon g \in G\} = \{\theta(a^k) \colon k \in \mathbb{Z}\}$. By Theorem 40(c), $\theta(G) = \{\theta(a)^k \colon k \in \mathbb{Z}\} = \langle \theta(a) \rangle$. So $\theta(G)$ is cyclic.

(b) Counterexample: take any $n \in \mathbb{N}$ and let $G = H = \mathbb{Z}_n$, and let $\theta\colon G \to H$, $[k] \mapsto [0]$ for $k = 0, 1, \ldots, n-1$. This is a homomorphism and $[1]$ is an element of G of order n, but $\theta(G) = \{[0]\}$ does not contain any element of order n since $o([0]) = 1$.

5. Let $\theta\colon G \to H$ be a homomorphism. For each $h \in H$, consider the preimage of h under θ, which is the set $P(h, \theta) = \{g \in G\colon \theta(g) = h\}$.

   (a) If $G = H = \mathbb{Z}_6$ and $\theta\colon G \to H$ is the mapping $\theta([k]) = [2k]$ for $0 \leq k < 6$, show that θ is a homomorphism. Then compute $\ker\theta$ and find $P(h, \theta)$ for every $h \in H$.

   (b) Now let G, H be any groups and let θ be any homomorphism $G \to H$.
   Let $K = \ker\theta$. Prove that for every $h \in H$,

$$\text{either } P(h,\theta) = \emptyset \quad \text{or} \quad P(h,\theta) = Ka \text{ for any } a \in P(h,\theta).$$

Solution (a) We have $\theta([k] \oplus [\ell]) = \theta([k+l]) = [2(k+\ell)]$ and $\theta([k]) \oplus \theta([\ell]) = [2k] \oplus [2\ell] = [2k+2\ell] = [2(k+\ell)]$. So θ is a homomorphism. We have $\ker\theta = \{[k]\colon \theta([k]) = e_{\mathbb{Z}_6} = [0]\} = \{[k]\colon [2k] = [0]\} = \{[0],[3]\}$ and so $P([0],\theta) = \ker\theta = \{[0],[3]\}$. Similar calculations give $P([1],\theta) = \emptyset$, $P([2],\theta) = \{[1],[4]\}$, $P([3],\theta) = \emptyset$, $P([4],\theta) = \{[2],[5]\}$ and $P([5],\theta) = \emptyset$.

(b) Let $h \in H$. If $P(h,\theta) \neq \emptyset$, choose $a \in P(h,\theta)$. Then $\theta(a) = h$.

We claim that $P(h,\theta) = Ka$.

Firstly, if $b \in P(h,\theta)$, then $\theta(b) = h = \theta(a)$, so $e_H = \theta(b)\theta(a)^{-1} = \theta(b)\theta(a^{-1}) = \theta(ba^{-1})$. Hence $ba^{-1} \in \ker\theta = K$, and $b = (ba^{-1})a \in Ka$. Hence $P(h,\theta) \subseteq Ka$.

Secondly, if $x \in Ka$ then $x = ka$ for some $k \in K$, so $\theta(x) = \theta(ka) = \theta(k)\theta(a) = e_H\theta(a) = \theta(a) = h$. So $x \in P(h,\theta)$. Hence $Ka \subseteq P(h,\theta)$.

So $P(h,\theta) = Ka$.

This proves that if $P(h,\theta) \neq \emptyset$ then $P(h,\theta) = Ka$ for every $a \in P(h,\theta)$. So we're done.