MA214 Sample paper, 2010

Sample solutions

① (a) $\alpha \in S_n$ is even if
α is the composition of an even number of transpositions.

So if α & β are even

then $\alpha = \alpha_1 \circ \dots \circ \alpha_k$ for some transpositions $\alpha_1, \dots, \alpha_k$

& $\beta = \beta_1 \circ \dots \circ \beta_m$ ——— " ——— $\beta_1, \dots, \beta_m$

where m, k are even

so $\alpha \circ \beta = \underbrace{\alpha_1 \circ \dots \circ \alpha_k \circ \beta_1 \circ \dots \circ \beta_m}_{\text{even no. } (m+k) \text{ of transpositions}}$ is even.

(b) $\gamma = \underbrace{\underbrace{(1284)}_{\text{odd}}\ \underbrace{\underbrace{(432)}_{\substack{\text{even}\\ \text{even}}}\ \underbrace{\underbrace{(57)}_{\text{odd}}\ \underbrace{(1423)}_{\text{odd}}}_{\text{even}}}_{\text{even}}}_{\text{odd}}$

No, γ is odd.

Or $\sigma(\gamma) = \sigma((1284))\,\sigma((432))\,\sigma((57))\,\sigma((1423))$
$= -1 \cdot 1 \cdot -1 \cdot -1 = -1$, so γ odd

①(c)

$$\gamma = (1284)(432)(57)(1423)$$
$$= (1284)(432)(1423)(57)$$
$$= \underline{(13284)(57)} \quad \text{(disjoint cycles)}$$

$$\gamma^k = ((13284)(57))^k$$
$$= (13284)^k (57)^k$$

So $\gamma^k = (1) \iff (13284)^k = (1)$
& $(57)^k = 1$
$\iff 5 \mid k$ & $2 \mid k$
$\iff 10 \mid k$.

So $o(\gamma)$ = smallest $k \geq 1$ with $\gamma^k = (1)$
= ——— " ——— $10 \mid k$

So $\underline{o(\gamma) = 10}$.

So $|H| = |\langle \gamma \rangle| = 10$.

① (d). $H \times H = \{ (\gamma^k, \gamma^l) : k, l \in \mathbb{Z} \}$.

<u>Claim</u> ~~$e(x)$~~ $x^{10} = e_{H\times H} \quad \forall x \in H \times H$.

<u>Pf</u> ~~Let~~ $x \in H \times H$

$\Rightarrow \quad x = (\gamma^k, \gamma^l)$, some $l \in \mathbb{Z}$

$$\Rightarrow \quad x^{10} = (\gamma^{10k}, \gamma^{10l}) = ((\gamma^{10})^k, (\gamma^{10})^l) = ((1)^k, (1)^l) = ((1), (1)) = e_{H\times H}.$$

So $\forall x \in H \times H : \quad o(x) \leq 10$

$\Rightarrow \forall x \in H \times H \quad o(x) < 100 = |H \times H|$

$\Rightarrow \quad H \times H \neq \langle x \rangle \quad \forall x \in H$

$\Rightarrow \quad H \times H$ not cyclic.

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② (a)

$G = D_n = \{ \iota, \rho, \rho^2, \dots, \rho^{n-1}, r, \rho \circ r, \rho^2 \circ r, \dots, \rho^{n-1} \circ r \}$.

(b) $\langle \rho \rangle = \{ \iota, \rho, \dots, \rho^{n-1} \}$ is a subgroup of order n

$\langle r \rangle = \{ \iota, r \}$ is a subgp. of order 2

$\langle \rho \circ r \rangle = \{ \iota, \rho \circ r \}$ is another.

(c). <u>REFLEXIVITY</u>: if $p \in P$
then $p = \iota(p)$
so $p \sim p$.

<u>SYMMETRY</u> if $p, q \in P$ & $p \sim q$

Then $\exists \alpha \in G : \alpha(p) = q$

so $p = \alpha^{-1}(q)$

so $\exists \beta \in G : \beta(q) = p$ [we take $\beta = \alpha^{-1}$]

so $q \sim p$.

② (c) cont'd

TRANSITIVITY: if $p, q, r \in P$

& $p \sim q$ & $q \sim r$,

then $\exists\, \alpha, \beta \in G$: $\alpha(p) = q$ & $\beta(q) = r$

so $\beta(\alpha(p)) = r$

so $\gamma(p) = r$ where $\gamma = \beta \circ \alpha \in G$ (since G is a group & $\alpha, \beta \in G$)

so $p \sim r$.

Hence $\sim$ is an equiv relation on P.

(d). We have $[p]_\sim = \{ q : \cancel{}\ p \sim q \}$

$= \{ q : \exists\, \alpha \in G : \alpha(p) = q \}$

so $\underline{[p]_\sim = \{ \alpha(p) : \alpha \in G \}}$

So $[\binom{1}{0}]_\sim = \{ \alpha(\binom{1}{0}) : \alpha \in G \}$

$\underset{(a)}{=} \{ \underbrace{\iota(\binom{1}{0}), \rho(\binom{1}{0}), \ldots, \rho^{n-1}(\binom{1}{0})}_{\text{the vertices of } Q_n},\ r(\binom{1}{0}), \rho \circ r(\binom{1}{0}), \ldots, \rho^{n-1} \circ r(\binom{1}{0}) \}$

$r(\binom{1}{0}) = \binom{1}{0}$, $\rho \circ r(\binom{1}{0}) = \rho(\binom{1}{0})$

& $\rho^k \circ r(\binom{1}{0}) = \rho^k(\binom{1}{0})$ just ... to write down the k^{th} vertex of Q_n

② (d) continued

So $[\binom{1}{0}]_\sim$ is the set of vertices of Q_n.

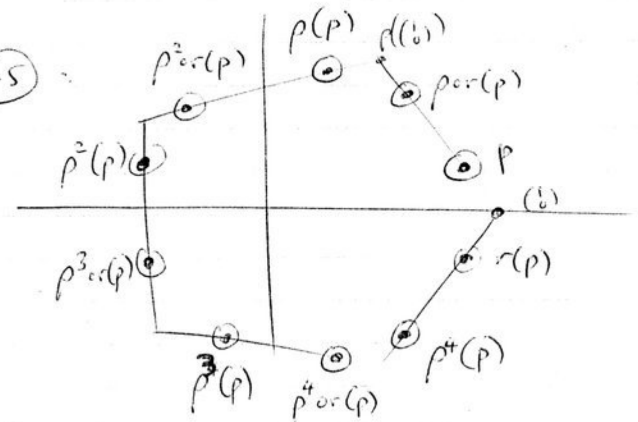
Let p be a point in Q_n other than a vertex or a midpoint of a side, eg $p = \frac{1}{3}\rho(\binom{1}{0}) + \frac{2}{3}\binom{1}{0}$

Then, clearly, $\iota(p), \rho(p), \rho^2(p), \ldots, \rho^{n-1}(p)$ are all different,

and $r(p), \rho(r(p)), \ldots, \rho^{n-1}(r(p))$ are all different and not in the first list

So $|[p]_\sim| = \left| \left\{ \begin{array}{l} \iota(p), \rho(p), \ldots, \rho^{n-1}(p), \\ r(p), \rho \circ r(p), \ldots, \rho^{n-1} \circ r(p) \end{array} \right\} \right| = 2n.$

eg (n=5):



& $[p]_\sim$ is the set of circled points; there are 10

③ (a) [This is bookwork.
You should come prepared
to give statements & proofs in
the exam if asked to do so]

(b). if $G \approx \mathbb{Z}_{29}$ then $|G| = |\mathbb{Z}_{29}| = 29$.

Conversely, if $|G| = 29$ then
since $|G|$ is prime,
we have: ~~G~~ G is a cyclic group ~~and~~ of order 29, by a Corollary of Lagrange's Thm.

Any finite cyclic group is isomorphic to $\mathbb{Z}_n$
where n is the order of the group.

So $G \approx \mathbb{Z}_{29}$.

③ (c) <u>Fund. Thm of Abelian Groups</u>:
Any finite abelian group G
is isomorphic to a direct product
$\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ where $n_1, \ldots, n_k$ are prime powers
with product $|G|$.

Also, if $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \approx \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_\ell}$
for some prime powers $m_1, \ldots, m_\ell$,
then ~~$(n_1, \ldots, n_k)$ is a~~ $\ell = k$, & $(m_1, \ldots, m_\ell)$ is a permutation of $(n_1, \ldots, n_k)$.

$996 = 2^2 \cdot 3 \cdot 83$ & 2, 3, 83 are prime.

So by this thm, ~~the~~ up to isomorphism, the abelian groups of order 996 are
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{83}$ & $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{83}$.

④ (a) A normal subgroup of G is a subgroup $N \leq G$ $\left[\begin{array}{l}\text{ie } N \subseteq G,\ N \neq \emptyset, \\ \& \ a,b \in N \Rightarrow ab^{-1} \in N\end{array}\right]$ such that

$$g \in G,\ n \in N \Rightarrow gng^{-1} \in N.$$

(b) (Bookwork).

(c). Let $H_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$

Then H is abelian & not cyclic.

Let $G = S_3 \times H$.

Then G is nonabelian

$$\left(\text{eg}/ \begin{array}{l} ((12), e_H)((13), e_H) = ((132), e_H) \\ \text{but } ((13), e_H)((12), e_H) = ((123), e_H) \end{array} \right)$$

~~and $N = \{(1)\}$~~

Let $\pi_H : G \to H$,

$$(\alpha, h) \longmapsto h \quad \text{for } (\alpha, h) \in G (= S_3 \times H);$$

Then π_H is a homomorphism, it's surjective, & $\ker \pi_H = S_3 \times \{e_H\} \trianglelefteq G$

Let $N = \ker \pi_H$.

Then $N \trianglelefteq G$ & $G/N \cong H$ by the Fund. Hom. Thm,

so G/N is abelian & not cyclic.

So the example: $G = S_3 \times H$ (where $H = \mathbb{Z}_2 \times \mathbb{Z}_2$)

$N = S_3 \times \{e_H\}$

will do.