UNIVERSITY OF DUBLIN

XMA1214

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF MATHEMATICS

JF Mathematics **Trinity Term 2010**
JF & SF Two Subject Moderatorship
SF Theoretical Physics

MA1214 — INTRODUCTION TO GROUP THEORY

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Credit will be given for the best 3 questions answered.

All questions have equal weight.

This sample is intended to give you some idea of the exam format,
but should *not* be used as a guide to its content.

This is a 2-hour exam.

1. Let $n \in \mathbb{N}$ and consider the group $(S_n, \circ)$.

   (a) [6 marks] Explain what it means to say that a permutation $\alpha \in S_n$ is *even*, and prove that if α are β are two even permutations in S_n, then $\alpha \circ \beta$ is also even.

   (b) [2 marks] Consider the permutation $\gamma = (1\ 2\ 8\ 4)(4\ 3\ 2)(5\ 7)(1\ 4\ 2\ 3)$ in S_8. Is γ an even permutation?

   (c) [6 marks] Write γ as a product of disjoint cycles, and determine the subgroup

   $$H = \langle \gamma \rangle$$

   of S_8.

   (d) [6 marks] Show that $H \times H$ is not a cyclic subgroup of $S_8 \times S_8$.

2. Let $P = \mathbb{R}^2 = \{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : x, y \in \mathbb{R} \}$ denote the plane. Fix an integer $n \geq 3$. Let Q_n be the regular polygon in P with n vertices on the unit circle, one of which is at $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Let $(G, \circ)$ be the symmetry group of Q_n, so that $G = D_n$, the dihedral group of order $2n$.

   Let $\iota \colon P \to P$ be the identity mapping, let $r \colon P \to P$ be given by reflection in the x-axis, and let $\rho \colon P \to P$ be the mapping of anticlockwise rotation by $2\pi/n$ radians.

   (a) [4 marks] List the elements of G, writing each element using one or more of the mappings ι, ρ and r.

   (b) [5 marks] Find a subgroup of G of order n, and two different subgroups of G of order 2.

   (c) [5 marks] Prove that the relation $\sim$ on P defined by

   $$p \sim q \iff \exists\, \alpha \in G \colon \alpha(p) = q$$

   is an equivalence relation.

   (d) [6 marks] Show that the equivalence class $[\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)]_\sim$ is the set of vertices of Q_n, and find an equivalence class containing $2n$ elements.

3. (a) [8 marks] State and prove Lagrange's theorem.

   (b) [6 marks] Show that if G is any group, then G is isomorphic to $\mathbb{Z}_{29}$ if and only if $|G| = 29$.

   (c) [6 marks] List all abelian groups of order 996, up to isomorphism. State the theorem you are using.

4. Let G be a group.

   (a) [3 marks] Explain what is meant by a *normal subgroup* of G.

   (b) [10 marks] If N is a normal subgroup of G, define the *quotient group* G/N. As part of this, you should give the definition of the usual operation on G/N and prove that it is a well-defined group operation.

   (c) [7 marks] Give an example of a non-abelian group G and a normal subgroup N of G such that G/N is abelian, but G/N is not cyclic. Explain why your answer is correct.