MA1214 exam questions and solutions, Trinity term 2010

1. Let $*$ be an operation on a set G. Suppose that $(G,*)$ is a group, and consider a second operation $\circledast$ on G, defined by

$$x \circledast y = x * y^{-1} \quad \text{for } x, y \in G$$

where, as usual, y^{-1} denotes the inverse of y with respect to the operation $*$.

(a) [3 marks] Explain what it means to say that $(G,*)$ is a *group*.

(b) [2 marks] Let e be the identity element of $(G,*)$. Compute $x \circledast e$ and $e \circledast x$.

(c) [8 marks] Prove that the following statements are equivalent:

(i). $(G,\circledast)$ is a group

(ii). $\circledast$ is associative

(iii). $x = x^{-1}$ for all $x \in G$

(iv). $* = \circledast$

[Hint: one way to do this is to give the proof in four steps, by first proving that (i) $\implies$ (ii), then that (ii) $\implies$ (iii), then that (iii) $\implies$ (iv), and finally that (iv) $\implies$ (i). You will be given marks for each step you successfully complete.]

(d) [4 marks] Use (c) to show that if $(G,\circledast)$ is a group, then $(G,*)$ is abelian.

(e) [3 marks] Give an example of a group $(G,*)$ of order 4 such that $(G,\circledast)$ is also a group, and explain why your answer is correct.

Solution (a) BOOKWORK $(G,*)$ is a group if and only if

- $*$ is an associative operation on G
- there is an element $e \in G$ which is an identity element for $*$
- every element of G is invertible with respect to $*$

(b) UNSEEN $x \circledast e = x * e^{-1} = x * e = e$ and $e \circledast x = e * x^{-1} = x^{-1}$.

(c) UNSEEN (i) $\implies$ (ii): If $(G,\circledast)$ is a group, then $\circledast$ is certainly associative (by the definition of a group).

(ii) $\implies$ (iii): if $\circledast$ is associative and $x \in G$, then

$$x^{-1} = e \circledast x = (e \circledast e) \circledast x = e \circledast (e \circledast x) = e \circledast x^{-1} = (x^{-1})^{-1} = x.$$

(iii) $\implies$ (iv): We have $y = y^{-1}$ for all $y \in G$, so $x * y = x * y^{-1} = x \circledast y$ for all $x, y \in G$, so $* = \circledast$.

(iv) $\implies$ (i): If $* = \circledast$ then $(G,*) = (G,\circledast)$, and $(G,*)$ is a group by hypothesis. So $(G,\circledast)$ is a group.

(d) UNSEEN If $x, y \in G$ and $(G,\circledast)$ is a group, then by (iii) we have $x * y = (x * y)^{-1} = y^{-1} * x^{-1} = y * x$. So $(G,*)$ is abelian.

(e) UNSEEN Every element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is equal to its own inverse. By (c), this does the job.

2. Let $P = \mathbb{R}^2 = \{\binom{x}{y} : x, y \in \mathbb{R}\}$ denote the plane. Fix an integer $n \geq 3$. Let Q_n be the regular polygon in P with n vertices on the unit circle, one of which is at $\binom{1}{0}$. Let $(G, \circ)$ be the symmetry group of Q_n, so that $G = D_n$, the dihedral group of order $2n$.

Let $\iota\colon P \to P$ be the identity mapping, let $r\colon P \to P$ be given by reflection in the x-axis, and let $\rho\colon P \to P$ be the mapping of anticlockwise rotation by $2\pi/n$ radians.

(a) [4 marks] List the elements of G, writing each element using one or more of the mappings ι, ρ and r.

(b) [8 marks] Let T be the intersection of Q_n with the x-axis.

(i). If n is even, what is $G_{(T)}$?

(ii). If n is odd, what is $G_{(T)}$?

Justify your answers.

(c) [4 marks] Let $H = \langle \rho \rangle$. Prove that H is a normal subgroup of G.

You may use the equation $r \circ \rho = \rho^{-1} \circ r$ without proof.

(d) [4 marks] Explain why the quotient group G/H is an abelian group of order 2.

Solution (a) BOOKWORK We have

$$G = D_n = \{\iota, \rho, \rho^2, \dots, \rho^{n-1}, r, \rho \circ r, \rho^2 \circ r, \dots, \rho^{n-1} \circ r\}.$$

(b) UNSEEN

(i). If n is even, then T contains two opposite vertices on the x-axis, so $r \in G_{(T)}$ (since r fixes every point on the x-axis) and $\rho_\pi = \rho^{n/2} \in G_{(T)}$. Clearly, no other rotation in G is in $G_{(T)}$, and since $G_{(T)}$ is a subgroup containing r, and $\rho^k \circ r \circ r = \rho^k$, the only other element apart from ι is $\rho^{n/2} \circ r$. So $G_{(T)} = \{\iota, \rho^{n/2}, r, \rho^{n/2} \circ r\}$.

(ii). If n is odd, then T contains $\binom{1}{0}$ and the midpoint p of the side of Q_n opposite $\binom{1}{0}$. No element of G maps p to $\binom{1}{0}$, since they are different distances from the origin, which is fixed by G. Hence $G_{(T)} = G_T$. The only elements of G fixing any point on the x-axis are ι and r, so $G_{(T)} = \{\iota, r\}$.

(c) BOOKWORK/HOMEWORK Let $H = \langle \rho \rangle$. Then H is the subgroup generated by ρ, so it's certainly a subgroup of G. Let $x \in G$. If $x \in H$ then $xHx^{-1} = H$, so suppose $x \notin H$. Then $x = \rho^k \circ r$ for some $k \in \mathbb{Z}$, so if $h \in H$, then $h = \rho^m$ for some $m \in \mathbb{Z}$, so $xhx^{-1} = \rho^k \circ r \circ \rho^m \circ r \circ \rho^{-k} = \rho^{k-m} \circ \rho^{-k} \in H$. Hence H is normal in G.

(d) UNSEEN Let $H = \langle \rho \rangle$. We have $|G| = |H| \cdot [G : H] = |H| \cdot |G/H|$ by Lagrange's theorem, so $|G/H| = |G|/|H| = 2n/n = 2$. Every group of order 2 is abelian. So G/H is abelian.

3. Let G be a finite group, and let us write the group operation on G as $(x, y) \mapsto xy$ and let e be the identity element of G. Let H be a subgroup of G and define a relation $\sim$ on G by

$$a \sim b \iff ab^{-1} \in H \quad \text{for } a, b \in G.$$

(a) [3 marks] Prove that $\sim$ is an equivalence relation on G.

(b) [3 marks] If $a \in G$, explain what is meant by the *right coset* Ha, and prove that $[a]_\sim = Ha$.

(c) [3 marks] Show that if $a \in G$ then $|Ha| = |H|$.

(d) [4 marks] Use a theorem about equivalence relations to deduce that $|H| \Bigm| |G|$. You should clearly state the theorem that you use.

(e) [7 marks] Show that if A and B are subgroups of G with $|A| = 35$ and $|B| = 44$, then $A \cap B = \{e\}$.

Solution

(a) BOOKWORK If $x \in G$ then $xx^{-1} = e \in H$, since H is a subgroup of G. So $x \sim x$.
If $x \sim y$ then $xy^{-1} \in H \implies y^{-1}x = (xy^{-1})^{-1} \in H \implies y \sim x$.
If $x \sim y$ and $y \sim z$ then $xy^{-1}, yz^{-1} \in H \implies xy^{-1}yz^{-1} = xz^{-1} \in H \implies x \sim z$.

(b) BOOKWORK $Ha = \{ha \colon h \in H\}$. If $b \in G$ then $b \in [a]_\sim \iff ab^{-1} \in H \iff a = hb$ for some $h \in H \iff a \in Hb$.

(c) BOOKWORK Let $\alpha \colon H \to Ha$, $h \mapsto hb$. Then α is one-to-one, since $h_1 a = h_2 a \iff h_1 = h_2 \iff h_1 b = h_2 b$, and it is clearly onto. So it is a bijection, so $|H| = |Ha|$.

(d) BOOKWORK The theorem says that the equivalence classes of an equivalence relation on a set form a partition of that set. So the equivalence classes $[a]_\sim = Ha$ partition G; since G is finite, this gives $G = Ha_1 \cup \cdots \cup Ha_k$ for some $a_1, \ldots, a_k \in G$. Since there are no overlaps in a partition and $|Ha_i| = |H|$, this gives $|G| = k|H|$. So $|H| \Bigm| |G|$.

(e) UNSEEN Since A and B are subgroups of G, their intersection $H = A \cap B$ is a subgroup of G (we proved this in an exercise). So H is a subgroup of A and of B. So $|H| \Bigm| |A| = 35 = 5 \cdot 7$ and $|H| \Bigm| |B| = 44 = 2^2 \cdot 11$. But the only integer dividing 35 and 44 is 1. So $|H| = 1$, so $H = \{e\}$.

4. Let G and H be groups, and let $\theta\colon G \to H$ be a mapping.

(a) [3 marks] What does it mean to say that θ is a *homomorphism*? What does it mean to say that θ is an *isomorphism*?

(b) [5 marks] Show that if θ is an isomorphism, then

$$G \text{ is abelian} \iff H \text{ is abelian.}$$

(c) [5 marks] By providing a counterexample, show that the conclusion of (b) may be false if θ is merely assumed to be a homomorphism.

(d) [7 marks] Suppose that θ is a surjective homomorphism. Prove that $K = \ker\theta$ is a normal subgroup of G, and that the quotient group G/K is isomorphic to H.

Solution (a) BOOKWORK θ is a homomorphism if $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in G$.
θ is an isomomorphism if θ is a bijective homomorphism.

(b) BOOKWORK Suppose that G is abelian. Then $\theta(a)\theta(b) = \theta(ab) = \theta(ba) = \theta(b)\theta(a)$. So $xy = yx$ for all $x, y \in \theta(G)$. Since $\theta(G) = H$, this shows that H is abelian.

Conversely, suppose that H is abelian. Since θ^{-1} is an isomorphism $H \to G$, the previous paragraph shows that G is abelian.

(c) UNSEEN Let $H = \{e\}$ and let $G = S_3$. Then G is not abelian, and $\theta\colon G \to H$, $\alpha \mapsto e$ is a homomorphism, but H is abelian.

(d) BOOKWORK If $s, t \in K$ then $\theta(st^{-1}) = \theta(s)\theta(t)^{-1} = ee^{-1} = e$, so $st^{-1} \in K$, so K is a subgroup of G. Also, if $k \in K$ then $\theta(k) = e$, so $\theta(gkg^{-1}) = \theta(g)\theta(k)\theta(g)^{-1} = \theta(g)\theta(g)^{-1} = e$. So K is a normal subgroup of G.

Let $\phi\colon G/K \to H$, $Kg \mapsto \theta(g)$. Then ϕ is well-defined, since $Kg_1 = Kg_2 \iff g_1g_2^{-1} \in K \iff \theta(g_1g_2^{-1}) = e \iff \theta(g_1) = \theta(g_2)$. This also shows that ϕ is injective, and it is surjective since θ is surjective. Finally, θ is a homomorphism, since $\theta((Kg_1)(Kg_2)) = \theta(Kg_1g_2) = \theta(g_1g_2) = \theta(g_1)\theta(g_2) = \phi(Kg_1)\phi(Kg_2)$. So θ is an isomorphism $G/K \to H$, so these groups are isomorphic.