MA114 Summer exam 2008

Sample solutions, Q1-4, 6(for 2), 7(i).

① (i) (Bookwork)

(ii) "f has a left inverse" means $(f: X \to Y)$

$$\exists\, g : Y \to X : \quad g \circ f = \iota_X$$

$\boxed{\Rightarrow}$ If f has ~~has~~ a left inverse $g: Y \to X$

~~then~~ & $x_1, x_2 \in X$ ~~with~~

then $f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

$$\Rightarrow (g\circ f)(x_1) = (g \circ f)(x_2)$$

$$\underset{(g\circ f = \iota_X)}{\Longrightarrow} x_1 = x_2$$

So f is injective.

$\boxed{\Leftarrow}$ If f is injective, suppose $X \neq \emptyset$ & let $x_0 \in X$.

then let $g: Y \to X$

be defined by $g(y) = \begin{cases} x_0 & \text{if } y \notin f(X) \\ x & \text{if } y = f(x) \text{ for some } x \in X. \end{cases}$

This is well-defined because ~~..~~ f is injective.

[Indeed, if $y = f(x_1) = f(x_2)$ then $x_1 = x_2$.]

And $g \circ f(x) = x$ for all $x \in X$.

So g is a left inverse for f.

[If $X = \emptyset$ then this statement is false unless $Y = \emptyset$ too]

(iii) $(f \circ (g \circ h))(x) = f(g(h(x))) = ((f\circ g)\circ h)(x)$,

whenever $x \in \text{domain}(h) = \text{domain}(f \circ (g \circ h))$
$= \text{domain}((f \circ g) \circ h)$.

So $f \circ (g \circ h) = (f \circ g) \circ h$.

(iv). ~~We have $g \circ f = \iota_X$ &~~

Suppose $f: X \to Y$ & $g, h: Y \to X$

with $g \circ f = \iota_X$ & $f \circ h = \iota_Y$.
("left inverse") ("right inverse")

~~Then~~

Then $g = g \circ \iota_Y = g \circ (f \circ h) = (g \circ f) \circ h$
$= \iota_X \circ h$
$= h$.

So $g = h$.

② (i) operation, group: homework

semigroup: don't worry about it

subgroup: homework.

(ii) $Z(G) = \{ h \in G : gh = hg \ \forall g \in G \}$

① $Z(G) \leq G$.

(SG0) $Z(G) \subseteq G$, by definition

& $e \in Z(G)$, clearly, so $Z(G) \neq \emptyset$.

(SG1) if $a, b \in Z(G)$

then $\forall g \in G : ga = ag$ & $gb = gb$

$\Rightarrow \forall g \in G :$ $gab = agb = abg$

$\Rightarrow ab \in Z(G)$.

So $a, b \in Z(G) \Rightarrow ab \in Z(G)$

(SG2) $a \in Z(G)$

$\Rightarrow \forall g \in G : ga = ag$

$\Rightarrow \forall g \in G : a^{-1}gaa^{-1} = a^{-1}aga^{-1}$

$\Rightarrow \forall g \in G : a^{-1}g = ga^{-1}$

$\Rightarrow a^{-1} \in Z(G)$

So $Z(G) \leq G$.

(Q2) Contd

② $Z(G)$ is abelian:

$a, b \in Z(G) \Rightarrow a \in Z(G), b \in G$

$\xrightarrow[\text{def of } Z(G)]{} ab = ba$.

So $Z(G)$ is abelian.

(iii) The $a^{th}$ row of the Cayley table consists of the elements $ax$ for $x \in G$.

if $y \in G$ then $y = a(a^{-1}y)$ is in this row.

~~If $y_1, y_2 \in G$ & $y_1 = ax_1$~~

If $y \in G$ & $y$ appears twice in this row

then $y = ax_1 = ax_2$ $(x_1, x_2 \in G, x_1 \neq x_2)$

$\Rightarrow x_1 = x_2$ by left cancellation, a contradiction.

So each $y \in G$ appears once, & only once, in this row.

(3)

(i) (a) Suppose $a^r \neq a^s$ & $r \neq s$.

We may suppose $r < s$ (else swap $r$ & $s$).

So $a^{s-r} = e$ & $s - r \in \mathbb{N}$

So $\{n \in \mathbb{N} : a^n = e\} \neq \emptyset$,

so it contains a least element by the least integer principle.

So there is a smallest $n > 0$ with $a^n = e$.

(b) Suppose $a^t = e$.

Division alg: write $t = nq + r$, $0 \leq r < n$

Then $a^{nq+r} = e$

$\Leftrightarrow (a^n)^q a^r = e$

$\Leftrightarrow a^r = e \underset{\substack{\text{defn of } n, \\ \& \ 0 \leq r < n}}{\Rightarrow} r = 0 \Leftrightarrow n | t$.

So $a^t = e \Rightarrow n | t$.

Conversely, $n | t \Rightarrow t = qn$, some $q \in \mathbb{Z}$

$\Rightarrow a^t = (a^n)^q = e^q = e$. ✓

(ii) <u>Fund. Thm of Ab Gps</u> (State it; bookwork)

$100 = 2^2 \cdot 5^2$

$$\left.\begin{array}{l} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} \\ \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\ \mathbb{Z}_4 \times \mathbb{Z}_{25} \end{array}\right\} \text{all ab. groups of order 100, up to isomorphism.}$$

(4) [This is all bookwork]

(5) [Q5 not on MA1214 syllabus]

(6) (i), (ii) [Bookwork for $\mathbb{Z}$]

(7) (i) [all bookwork]

# MA114 Summer 2009 Sample Solutions

(1) (a) Bookwork

(b). (i)

$$\begin{aligned}\alpha(A\cup B) &= \{\alpha(x) : x\in A\cup B\}\\ &= \{\alpha(x) : x\in A \text{ or } x\in B\}\\ &= \{\alpha(x) : x\in A\}\cup\{\alpha(x) : x\in B\}\\ &= \alpha(A)\cup\alpha(B).\end{aligned}$$

(ii) $\alpha(A\cap B) = \{\alpha(x) : x\in A\cap B\}$

so

$$\begin{aligned} y\in\alpha(A\cap B) &\iff \exists\, x\in A\cap B : y=\alpha(x)\\ &\Rightarrow \exists\, x\in A : y=\alpha(x) \;\&\; \exists\, x'\in B : y=\alpha(x') \quad (\text{take } x'=x)\\ &\iff y\in\alpha(A)\cap\alpha(B)\end{aligned}$$

(c). Let $S=U=\{1\}$, $T=\{1,2\}$,

$$\delta(1)=\delta(2)=1,\quad \gamma(1)=1$$

Then $\delta\circ\gamma = 1_S$ is invertible, but neither $\delta$ nor $\gamma$ are.