Tutorial 4: Miscellanea.

Exercise 1. Give an algorithm for constructing the character table of the symmetric group S_{n-1} out of that of S_n, using the restriction of representations.

Exercise 2. Using the Hook length formula, prove that, for a Specht representation $V_\lambda \in \mathrm{Irrep}(S_n)$, one has $\dim_{\mathbb{C}}(V_\lambda) < n$ if and only if λ belongs either to one of the four families $V_n, V_{1^n}, V_{n-1,1}, V_{2,1^{n-2}}$, or to the list of small exceptions $V_{2,2}, V_{2,2,2}, V_{3,3}$.

Exercise 3. Using representation theory, prove the

Fundamental Theorem of Finite Abelian Groups: Every finite abelian group is isomorphic to a direct product of cyclic groups of prime-power order, where the decomposition is unique up to the order in which the factors are written.

Exercise 4. Recall that for any $n \in \mathbb{N}$, the vector space $\mathbb{C}^n$ can be seen as a representation of the group $\mathrm{GL}_n = \mathrm{Mat}^*_{n\times n}(\mathbb{C})$, where a matrix acts on a vector by usual multiplication. Without using the Schur-Weyl duality, show that the symmetric square $S^2(\mathbb{C}^n)$ and the alternating square $\Lambda^2(\mathbb{C}^n)$ of this representation are irreducible GL_n-reps.

Exercise 5. For any $n \geq 3$, construct the character table of the *dihedral group* D_{2n} of symmetries of a regular n-gon.

Exercise 6. Let G be a finite group. Consider the vector space $V := \mathbb{C}G = \bigoplus_{g\in G} \mathbb{C}e_g$, and the linear map

$$\sigma\colon V\otimes V \to V\otimes V,$$
$$e_g \otimes e_h \mapsto e_{ghg^{-1}} \otimes e_g,$$

where $g, h \in G$. Show that σ satisfies the Yang–Baxter equation. Write explicitly the associated representation of the braid group B_n.

Exercise 7. Consider two groups G and H. Prove that the direct product $G \times H$ of the underlying sets is also a group, with the following multiplication:

$$(g, h) \cdot (g', h') = (gg', hh').$$

From now on, by $G \times H$ we will mean this direct product group. Show that, for any $(V, \rho) \in \mathrm{Rep}(G)$ and $(W, \pi) \in \mathrm{Rep}(H)$, $(V \otimes W, \rho\otimes\pi) \in \mathrm{Rep}(G \times H)$, where $(\rho\otimes\pi)(g, h) = \rho(g)\otimes\pi(h)$. Further, supposing G and H finite, explain why this construction yields a bijection

$$\mathrm{Irrep}(G) \times \mathrm{Irrep}(H) \xleftrightarrow{1:1} \mathrm{Irrep}(G \times H),$$
$$((V, \rho)\,,\,(W, \pi)) \longmapsto (V \otimes W, \rho\otimes\pi).$$