Tutorial 3: Tensor products of representations. Representations of S_n and Young diagrams.

Exercise 1. Our aim is the following isomorphism of S_n-representations, $n \geq 3$:

$$V_{n-1,1} \cong V^{st}.$$

1. For $1 \leq i \leq n$, denote by T_i the Young tableau of shape $(n-1,1)$ having i in the second row, and the remaining numbers from $\{1,\ldots,n\}$ arranged in the increasing order in the first row. Show that their classes $[T_i]$ form a complete list of Young tabloids of shape $(n-1,1)$, without repetition.

   Solution. Young tabloids of shape $(n-1,1)$ are Young tableaux of shape $(n-1,1)$, considered up to row permutation. Permutations inside the first row can always arrange the entries of this row in the increasing order, so every Young tableau belongs to one of the classes $[T_i]$. Further, these classes are pairwise disjoint, since the row permutations do not affect the entry in the second row.

2. Recall that a permutation $\sigma \in S_n$ acts on a Young tableau by acting on each of its cells. Compute $\sigma \cdot [T_i]$ for any $1 \leq i \leq n$.

   Solution. Under the action of σ, the entry i of the second row of T_i becomes $\sigma(i)$, so
   $$\sigma \cdot [T_i] = [\sigma \cdot T_i] = [T_{\sigma(i)}].$$

3. Identify the above S_n-representation structure on the vector space $\mathbb{C}M_{n-1,1}$ of linear combinations of shape $(n-1,1)$ Young tabloids with a familiar representation.

   Solution. The S_n-representation $\mathbb{C}M_{n-1,1}$ is isomorphic to the permutation representation $V^{perm} = \oplus_{i=1}^n e_i$, $\sigma \cdot e_i = e_{\sigma(i)}$. The corresponding isomorphism sends $\sum_i \alpha_i e_{[T_i]}$ to $\sum_i \alpha_i e_i$, where $\alpha_i \in \mathbb{C}$.

4. Determine the column groups C_{T_j} of all the Young tableaux T_j.

   Solution. The first column of T_j consists of two entries $(1,j)$ if $j \neq 1$, and $(2,1)$ if $j=1$. So $C_{T_j} = \{\mathrm{Id}, (1j)\}$ or $\{\mathrm{Id}, (12)\}$ respectively.

5. Express the Young polytabloids $\mathcal{E}_{T_j}$ in terms of the basis $(e_{[T_i]})_{1 \leq i \leq n}$ of $\mathbb{C}M_{n-1,1}$.

   Solution. If $j \neq 1$, then one computes
   $$\mathcal{E}_{T_j} = \sum_{\sigma \in C_{T_j}} \mathbf{sgn}(\sigma)\sigma \cdot e_{[T_j]} = \mathrm{Id} \cdot e_{[T_j]} - (1j) \cdot e_{[T_j]} = e_{[T_j]} - e_{[T_1]}.$$
   Similarly, $\mathcal{E}_{T_1} = e_{[T_1]} - e_{[T_2]}$.

6. Show that the $\mathcal{E}_{T_j}$ span a sub-representation of $\mathbb{C}M_{n-1,1}$ isomorphic to the standard representation of S_n.

   Solution. The S_n-representation isomorphism from Q3 identifies the span of the $\mathcal{E}_{T_j}$, explicitly presented in Q5, with the span of the $e_j - e_1$ in V^{perm}, which, by definition, is the standard representation V^{st} of S_n.

Exercise 2. Representations of S_6

1. Write down all partitions of 6.

Solution.

$$\begin{array}{llll} 6=6, & 6=4+1+1, & 6=3+1+1+1, & 6=2+1+1+1+1, \\ 6=5+1, & 6=3+3, & 6=2+2+2, & 6=1+1+1+1+1+1. \\ 6=4+2, & 6=3+2+1, & 6=2+2+1+1, & \end{array}$$

2. How many irreps does S_6 have?

Solution. Recall that the conjugacy classes of S_n assemble permutations with the same cycle type, and these cycle types are encoded by partitions of n. So $\#\operatorname{Irrep}(S_6) = \#\operatorname{Conj}(S_6) = \#\{\lambda \vdash 6\} = 11$.

3. To what partitions do the irreps V^{tr}, $V^{\mathbf{sgn}}$, V^{st}, and $V^{st} \otimes V^{\mathbf{sgn}}$ correspond?

Solution. As we saw in class and in Exercise 1 above,

$$V^{tr} \cong V_6, \qquad V^{\mathbf{sgn}} \cong V_{1^6}, \qquad V^{st} \cong V_{5,1}, \qquad (V^{st})' := V^{st} \otimes V^{\mathbf{sgn}} \cong V_{2,1^4}.$$

4. Write down the characters of these four irreps.

Solution. Recall that $\chi^{V^{\mathbf{sgn}}}(\sigma) = 1$ for even permutations σ, and -1 for odd σ. Further, $\chi^{V^{st}}(\sigma) = \#\{1,\ldots,n\}^\sigma - 1$, where $\#\{1,\ldots,n\}^\sigma$ is the number of fixed points for σ, and $\chi^{V^{st}\otimes V^{\mathbf{sgn}}}(\sigma) = \chi^{V^{st}}(\sigma)\chi^{V^{\mathbf{sgn}}}(\sigma)$. Finally, the sizes of the conjugacy classes of S_6 can be computed by Theorem 9 from Lecture 15. This yields the first four columns of the character table for S_6:

$\#\mathcal{C}$	1	15	40	90	144	120	45	120	90	40	15
$\lambda \vdash 6$	1^6	21^4	31^3	41^2	51	6	2^21^2	321	42	3^2	2^3
V^{tr}	1	1	1	1	1	1	1	1	1	1	1
$V^{\mathbf{sgn}}$	1	-1	1	-1	1	-1	1	-1	1	1	-1
V^{st}	5	3	2	1	0	-1	1	0	-1	-1	-1
$(V^{st})'$	5	-3	2	-1	0	1	1	0	-1	-1	1

5. Determine the degree of the irrep $V_{2,2,2}$ using two methods:
   (a) first, by counting standard Young tableaux;
   (b) then, by the Hook length formula.

Solution. In a standard Young tableau of shape $(2,2,2)$, the entry 1 has to lie in the top left cell, and the entry 6 in the bottom right cell. The restrictions on the remaining cells are $c_{2,1} < c_{3,1}$, $c_{2,1} < c_{2,2}$ and $c_{1,2} < c_{2,2}$. Here $c_{i,j}$ denotes the entry in the cell (i,j) of our Young tableau. There are $\binom{4}{2} = 6$ ways of splitting the numbers $\{2,3,4,5\}$ into two ordered pairs, corresponding to the two columns. The remaining condition $c_{2,1} < c_{2,2}$ is violated for such two pairs if and only if $c_{1,2} < c_{2,2} < c_{2,1} < c_{3,1}$, which means $(c_{2,1}, c_{3,1}) = (4,5)$. Overall, one gets $6-1=5$ standard Young tableaux of shape $(2,2,2)$, which we will index by the non-unit entries of the first column:

$$T_{2,3} = \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}, \quad T_{2,4} = \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array}, \quad T_{3,4} = \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&6\\\hline\end{array}, \quad T_{2,5} = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array}, \quad T_{3,5} = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}.$$

So, $\dim_{\mathbb{C}} V_{2,2,2} = \#\,\mathrm{SYT}_{2,2,2} = 5$.

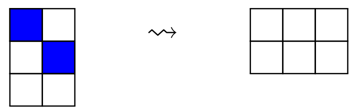
Alternatively, the Hook length formula gives

$$\dim_{\mathbb{C}} V_{2,2,2} = \frac{6!}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 5,$$

where the hook lengths of the cells are $\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$.

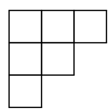
6. Identify the representation $V_{2,2,2} \otimes V^{\mathbf{sgn}}$.

   Solution. $V_{2,2,2} \otimes V^{\mathbf{sgn}} \cong V_{(2,2,2)^t} = V_{3,3}$, where the conjugate partition $(2,2,2)^t = (3,3)$ of $(2,2,2)$ is obtained by reflection along the main diagonal (highlighted in blue):

7. For which irreps V of S_6 does one have an isomorphism of representations $V \cong V \otimes V^{\mathbf{sgn}}$? Give the answer in terms of Specht irreps.

   Solution. We know that $V_\lambda \otimes V^{\mathbf{sgn}} \cong V_{\lambda^t}$, so $V_\lambda \cong V_\lambda \otimes V^{\mathbf{sgn}}$ is equivalent to $V_\lambda \cong V_{\lambda^t}$, which means that the partition λ is self-conjugate, $\lambda = \lambda^t$. There is only one self-conjugate partition of S_6:

    $\longleftrightarrow$ $V_{3,2,1}$.

8. Determine the degrees of all irreps of S_6 using your favourite method(s).

   Solution. Here are the degrees of the irreps not taken care of by Q3:

   $\dim_{\mathbb{C}} V_{2,2,2} = \dim_{\mathbb{C}} V_{3,3} = 5,$ $\qquad$ $\dim_{\mathbb{C}} V_{3,2,1} = 16,$

   $\dim_{\mathbb{C}} V_{4,1,1} = \dim_{\mathbb{C}} V_{3,1,1,1} = 10,$ $\qquad$ $\dim_{\mathbb{C}} V_{4,2} = \dim_{\mathbb{C}} V_{2,2,1,1} = 9.$

9. Check that the sum of the squares of these degrees takes the expected value.

   Solution. As expected,

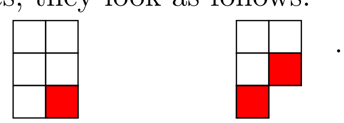
   $$1^2 + 1^2 + 5^2 + 5^2 + 5^2 + 5^2 + 16^2 + 10^2 + 10^2 + 9^2 + 9^2 = 720 = \#S_6.$$

10. Recall the classical symmetric group inclusions $\iota_n \colon S_n \to S_{n+1}$, given by

    $$\iota_n(\sigma)(k) = \begin{cases} \sigma(k), & k \leq n; \\ n+1, & k = n+1. \end{cases}$$

    Decompose into irreps the representations $\iota_5^*(V_{2,2,2})$ of S_5, and $\iota_4^*\iota_5^*(V_{2,2,2})$ of S_4.

    Solution. As seen in class, such decompositions are determined by the "removable" squares of Young diagrams. In our cases, they look as follows:

    So, $\iota_5^*(V_{2,2,2}) \cong V_{2,2,1}$, and $\iota_4^*\iota_5^*(V_{2,2,2}) \cong \iota_4^* V_{2,2,1} \cong V_{2,2} \oplus V_{2,1,1}$.

11. Compute the value of the character χ of $V_{2,2,2}$ on Id, (23), and $(23)(45)$. Show that $\chi((23)(45)(16)) \in \{3,5\}$. (*Hint:* Use the explicit basis for $V_{2,2,2}$ given by the Young polytabloids of the standard Young tableaux of shape $(2,2,2)$.)

Solution. As usual, $\chi(\mathrm{Id}) = \dim_{\mathbb{C}} V_{2,2,2} = 5$. On other permutations, the evaluation of χ is much more delicate. In class we have proved that

$$\sigma \cdot \mathcal{E}_T = \mathcal{E}_{\sigma \cdot T}.$$

Further, for $\sigma \in C_T$, one has $C_T = \sigma C_T$, so $\sigma \cdot \mathcal{E}_T = \mathbf{sgn}(\sigma)\mathcal{E}_T$. So, if for a $T \in \mathrm{SYT}_{2,2,2}$, a column permutation transforms $\sigma \cdot T$ into a standard Young tableau, we are fine. Otherwise we should express $\mathcal{E}_{\sigma \cdot T}$ as a linear combination of $\mathcal{E}_{T_i}$ for $T_i \in \mathrm{SYT}_{2,2,2}$, which is very hard in practice.

Let us now look how things work for $\sigma = (23)$, $(23)(45)$, or $(23)(45)(16)$.

$$(23) \cdot T_{2,3} = \begin{array}{|c|c|}\hline 1&4\\\hline 3&5\\\hline 2&6\\\hline\end{array} \overset{(23)}{\longmapsto} \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array} = T_{2,3}, \qquad (23) \cdot T_{2,4} = \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&6\\\hline\end{array} = T_{3,4},$$

$$(23) \cdot T_{3,4} = \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array} = T_{2,4}, \qquad (23) \cdot T_{2,5} = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array} = T_{3,5}, \qquad (23) \cdot T_{3,5} = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array} = T_{2,5}.$$

Thus (12) permutes $\mathcal{E}_{T_{2,4}}$ and $\mathcal{E}_{T_{3,4}}$, and $\mathcal{E}_{T_{2,5}}$ and $\mathcal{E}_{T_{3,5}}$. Moreover,

$$(23) \cdot \mathcal{E}_{T_{2,3}} = \mathbf{sgn}((23))\mathcal{E}_{T_{2,3}} = -\mathcal{E}_{T_{2,3}}.$$

Summarising,

$$\chi((23)) = \mathrm{tr}\begin{pmatrix} -1&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix} = -1.$$

Similarly,

$$(23)(45) \cdot T_{2,3} = \begin{array}{|c|c|}\hline 1&5\\\hline 3&4\\\hline 2&6\\\hline\end{array} \overset{(23)(45)}{\longmapsto} \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array} = T_{2,3}, \qquad (23)(45) \cdot T_{2,4} = T_{3,5},$$

$$(23)(45) \cdot T_{3,4} = T_{2,5}, \qquad (23)(45) \cdot T_{2,5} = T_{3,4}, \qquad (23)(45) \cdot T_{3,5} = T_{2,4},$$

and $\chi((23)(45)) = \mathbf{sgn}((23)(45)) = 1$.

Finally, since Young tabloids are considered only up to row permutations, and a permutation of two rows does not change the column group, one has $\mathcal{E}_T = \mathcal{E}_{\overline{T}}$, where $\overline{T}$ consists of the second column of T followed by the first one. (CAUTION: If you permute only some entries of the two columns, the corresponding Young polytabloid might change!) Using this, one computes

$$(23)(45)(16) \cdot T_{2,4} = \begin{array}{|c|c|}\hline 6&2\\\hline 3&4\\\hline 5&1\\\hline\end{array} \overset{-}{\longmapsto} \begin{array}{|c|c|}\hline 2&6\\\hline 4&3\\\hline 1&5\\\hline\end{array} \overset{(214)(356)}{\longmapsto} \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array} = T_{2,4},$$

so $(23)(45)(16) \cdot \mathcal{E}_{T_{2,4}} = \mathbf{sgn}((214)(356))\mathcal{E}_{T_{2,4}} = \mathcal{E}_{T_{2,4}}$. The same reasoning shows that $(23)(45)(16)$ acts trivially on $T_{3,4}$, $T_{2,5}$, and $T_{3,5}$. So, in a carefully chosen basis, the matrix of $(23)(45)(16)$ has the form

$$M = \begin{pmatrix} 1&0&0&0&*\\ 0&1&0&0&*\\ 0&0&1&0&*\\ 0&0&0&1&*\\ 0&0&0&0&a \end{pmatrix}.$$

From $((23)(45)(16))^2 = \mathrm{Id}$ follows $M^2 = I_5$, hence $a = \pm 1$, and $\chi((23)(45)(16)) \in \{3, 5\}$.

12. Using computations from Question 11, give a new proof of the fact that the irreps $V_{2,2,2}$ and $V_{2,2,2} \otimes V^{\mathbf{sgn}}$ are non-isomorphic.

Solution. One can compare the characters of these representations evaluated on $(23) \in S_6$:

$$\chi^{V_{2,2,2}}((23)) = -1 \neq 1 = (-1) \cdot (-1) = \chi^{V_{2,2,2}}((23))\chi^{V^{\mathbf{sgn}}}((23)) = \chi^{V_{2,2,2} \otimes V^{\mathbf{sgn}}}((23)).$$

13. What are the degrees of the reps $\Lambda^2(V^{st}), S^2(V^{st}), \Lambda^2(V_{2,2,2}), S^2(V_{2,2,2})$?

Solution. As was shown in class, for any representation V of degree d,

$$\dim_{\mathbb{C}} \Lambda^2(V) = \frac{1}{2}(d^2 - d), \qquad \dim_{\mathbb{C}} S^2(V) = \frac{1}{2}(d^2 + d).$$

This gives $\dim_{\mathbb{C}} \Lambda^2(V^{st}) = \dim_{\mathbb{C}} \Lambda^2(V_{2,2,2}) = 10$, $\dim_{\mathbb{C}} S^2(V^{st}) = \dim_{\mathbb{C}} S^2(V_{2,2,2}) = 15$.

14. Compute the value of the characters of $\Lambda^2(V^{st}), S^2(V^{st}), \Lambda^2(V_{2,2,2})$, and $S^2(V_{2,2,2})$, on Id, (23), (23)(45), and (23)(45)(16). (For the last permutation, it suffices to give two possible values.)

Solution. As was shown in class, for any G-representation V of degree d and any $g \in G$,

$$\chi^{\Lambda^2(V)}(g) = \frac{1}{2}(\chi^V(g)^2 - \chi^V(g^2)), \qquad \chi^{S^2(V)}(g) = \frac{1}{2}(\chi^V(g)^2 + \chi^V(g^2)).$$

Our permutations Id, (23), (23)(45), and (23)(45)(16) all square to Id. The desired character values then have the following values (the values of some of the irreps of S_6 were recalled in the table for reference in further questions):

$\mathcal{C}$	[Id]	[(23)]	[(23)(45)]	[(23)(45)(16)]
V^{tr}	1	1	1	1
$V^{\mathbf{sgn}}$	1	-1	1	-1
V^{st}	5	3	1	-1
$(V^{st})'$	5	-3	1	1
$V_{2,2,2}$	5	-1	1	3 or 5
$V'_{2,2,2}$	5	1	1	-3 or -5
$\Lambda^2(V^{st})$	10	2	-2	-2
$(\Lambda^2(V^{st}))'$	10	-2	-2	2
$S^2(V^{st})$	15	7	3	3
$\Lambda^2(V_{2,2,2})$	10	-2	-2	2 or 10
$S^2(V_{2,2,2})$	15	3	3	7 or 15

15. Deduce from these computations that each of the four reps from the previous point has an irreducible direct summand of degree 9 or 10.

Solution. We saw that S_6 has irreps in degrees 1, 5, 9, 10, and 16 only. Our four reps are two small to have irreducible summands of degree 16. If they did not have any irreducible summands of degree 9 nor 10, this would mean that they are direct sums of degree 1 and 5 irreps. For $\Lambda^2(V^{st})$ and $\Lambda^2(V_{2,2,2})$, this directly leads to contradiction: their characters take the value -2 on (23)(45), while, as shown in the table above, the characters of all degree 1 and 5 irreps are positive on (23)(45).
For $S^2(V^{st})$ and $S^2(V_{2,2,2})$, the same argument shows that they both have to be direct sums of 3 degree 5 irreps. Looking at the character values on (23), the only possibility for $S^2(V^{st})$ is $S^2(V^{st}) \cong 2V^{st} \oplus V'_{2,2,2}$, but this is not compatible with the character values on (23)(45)(16). Looking at the character values on (23)(45)(16), the only possibilities for $S^2(V_{2,2,2})$ are $S^2(V_{2,2,2}) \cong 3V_{2,2,2}$, or $S^2(V_{2,2,2}) \cong 2V_{2,2,2} \oplus V^{tr}$, or $S^2(V_{2,2,2}) \cong 2V_{2,2,2} \oplus (V^{st})'$. Neither of them is compatible with the character values on (23).

16. How would you check that $\Lambda^2(V^{st})$ is an irrep? (You do not have to carry out the computation. Simply explain what formulas and properties you would use.) From now on, you can assume it.

Solution. According to the irreducibility criterion seen in class, it suffices to check that $(\chi^{\Lambda^2(V^{st})}(g), \chi^{\Lambda^2(V^{st})}(g)) = 1$. The computation of the character of $\Lambda^2(V^{st})$ was outlined in Q14.

17. Describe all irreps of degree 10.

    Solution. As seen in Q8, there are exactly two of them. In the language of Specht reps, they are $V_{4,1,1}$ and $V_{3,1,1,1} \cong V_{4,1,1} \otimes V^{\mathbf{sgn}}$. But the previous question allows us to describe them more explicitly as $\Lambda^2(V^{st})$ and $\Lambda^2(V^{st}) \otimes V^{\mathbf{sgn}}$. We don't have enough information to tell which is which.

18. Check that $S^2(V^{st})$ contains an irreducible direct summand of degree 9. (*Hint:* Use Questions 15 and 17, and evaluate the characters for possible decompositions of $S^2(V^{st})$ on (23)(45), then on (23), then on (123).)

    Solution. By Q15 and Q17, if $S^2(V^{st})$ does not contain an irreducible direct summand of degree 9, then it contains an irreducible degree 10 summand, which has to be $\Lambda^2(V^{st})$ or $(\Lambda^2(V^{st}))' := \Lambda^2(V^{st}) \otimes V^{\mathbf{sgn}}$. Its complement in $S^2(V^{st})$ is either a degree 5 summand, or a direct sum of 5 degree 1 summands. Looking at the character values on (23)(45) (see the table in the proof of Q14), one eliminates the case of a degree 5 complement. Thus, $S^2(V^{st}) \cong V \oplus kV^{tr} \oplus (5-k)V^{\mathbf{sgn}}$, where $V \in \{\Lambda^2(V^{st}), (\Lambda^2(V^{st}))'\}$, and $k \in \{0, 1, \ldots, 5\}$. For $V = (\Lambda^2(V^{st}))'$, character evaluation on (23) yields $7 = -2+k-(5-k)$, so $k = 7$, which is impossible. For $V = \Lambda^2(V^{st})$, character evaluation on (23) yields $7 = 2 + k - (5 - k)$, so $k = 5$, and $S^2(V^{st}) \cong \Lambda^2(V^{st}) \oplus 5V^{tr}$. Using the formulas from Q14, this translates as $\chi^{V^{st}}(\sigma^2) = 5\chi^{V^{tr}}(\sigma) = 5$ for all $\sigma \in S_6$. But this is false, for example, for $\sigma = (123)$.

19. Show that the second irrep of degree 9 can be found inside $S^2(V^{st}) \otimes V^{\mathbf{sgn}}$.

    Solution. We have just established the decomposition $S^2(V^{st}) \cong V \oplus W$, where V is a degree 9 irrep (by Q8, it is either $V_{4,2}$ or $V_{2,2,1,1}$), and W is a direct sum of lower degree irreps. By Q8, the second degree 9 irrep is $V \otimes V^{\mathbf{sgn}}$. But

    $$S^2(V^{st}) \otimes V^{\mathbf{sgn}} \cong (V \oplus W) \otimes V^{\mathbf{sgn}} \cong V \otimes V^{\mathbf{sgn}} \oplus W \otimes V^{\mathbf{sgn}},$$

    so $V \otimes V^{\mathbf{sgn}}$ is indeed an irreducible direct summand of $S^2(V^{st}) \otimes V^{\mathbf{sgn}}$.

Remark. In HW3, you'll generalise some of the phenomena observed here for S_6, and show the following S_n-representation isomorphisms:

$$\Lambda^2(V^{st}) \cong V_{n-2,1,1}, \qquad S^2(V^{st}) \cong V_{n-2,2} \oplus V^{st} \oplus V^{tr}.$$

Exercise 3. Our aim is to show that, given a faithful representation (V, ρ) of a finite group G, any irrep W of G is contained in some of the tensor powers $V^{\otimes n}$. Recall that

- a representation $\rho\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is called *faithful* if the map ρ is injective;
- the *tensor powers* $V^{\otimes n}$ are defined as $(\ldots((V \otimes V) \otimes V)\ldots) \otimes V$, with n copies of V; by $V^{\otimes 0}$ we mean the trivial representation.

1. Express the character $\chi^{V^{\otimes n}}$ in terms of χ^V.

    Solution. Since, by definition, $V^{\otimes n} = V^{\otimes(n-1)} \otimes V$, one has $\chi^{V^{\otimes n}}(g) = \chi^{V^{\otimes(n-1)}}(g)\chi^V(g)$. By induction, one concludes that $\chi^{V^{\otimes n}}(g) = (\chi^V(g))^n$ for all $n \geq 1$ and $g \in G$.

2. For any representations V and W of a finite group G, prove the following equality of formal power sums:
$$\sum_{n\geq 0}((\chi^V)^n, \chi^W)t^n = \frac{1}{\#G}\sum_{\mathcal{C}\in\mathrm{Conj}(G)}\frac{\#\mathcal{C}\,\overline{\chi^W(\mathcal{C})}}{1-\chi^V(\mathcal{C})t}.$$
Here $\chi(\mathcal{C})$ is defined as $\chi(g)$ for any $g\in\mathcal{C}$.

Solution.
$$\#G\sum_{n\geq 0}((\chi^V)^n, \chi^W)t^n = \sum_{n\geq 0}(t^n\sum_{g\in G}\chi^V(g)^n\,\overline{\chi^W(g)}) = \sum_{n\geq 0}(t^n\sum_{\mathcal{C}\in\mathrm{Conj}(G)}(\#\mathcal{C}\,\chi^V(\mathcal{C})^n\,\overline{\chi^W(\mathcal{C})}))$$
$$= \sum_{\mathcal{C}\in\mathrm{Conj}(G)}(\#\mathcal{C}\,\overline{\chi^W(\mathcal{C})}\sum_{n\geq 0}(t\,\chi^V(\mathcal{C}))^n) = \sum_{\mathcal{C}\in\mathrm{Conj}(G)}(\#\mathcal{C}\,\overline{\chi^W(\mathcal{C})}\frac{1}{1-\chi^V(\mathcal{C})t}).$$
We had the right to interchange the two summation signs since the sum $\sum_{\mathcal{C}\in\mathrm{Conj}(G)}$ is finite.

3. Now suppose V faithful, and W irreducible and not contained in any $V^{\otimes n}$. Show that all coefficients of the formal power series $\sum_{n\geq 0}((\chi^V)^n, \chi^W)t^n$ are then zero.

   Solution. By Q1, $(\chi^V)^n = \chi^{V^{\otimes n}}$. Since the multiplicity of the irrep W in $V^{\otimes n}$ is 0
$$((\chi^V)^n, \chi^W) = (\chi^{V^{\otimes n}}, \chi^W) = 0.$$

4. Prove that $\chi^V(\mathcal{C}) = \dim_{\mathbb{C}}(V)$ if and only if $\mathcal{C}$ is the class of 1.

   Solution. Take any $g\in\mathcal{C}$. Put $d=\dim_{\mathbb{C}}(V)$. We know that, in a suitable basis, the matrix of $\rho(g)$ is diagonal, with d roots of unity $\theta_1,\ldots,\theta_d$ on the diagonal. Since the θ_i are roots of unity, $\mathrm{Re}(\theta_i)\leq 1$, and $\mathrm{Re}(\theta_i)=1$ if and only if $\theta_i=1$. Now, $\chi^V(\mathcal{C})=\sum_i\theta_i$ is d iff $\sum_i\mathrm{Re}(\theta_i)=d$ and $\sum_i\mathrm{Im}(\theta_i)=0$, which is then equivalent to all $\theta_i=1$. But this means $\rho(g)=\mathrm{Id}$. Since our representation is faithful, this is equivalent to $g=1$.

5. Deduce from this that $\sum_{\mathcal{C}\in\mathrm{Conj}(G)}\frac{\#\mathcal{C}\,\overline{\chi^W(\mathcal{C})}}{1-\chi^V(\mathcal{C})t}$ cannot be the zero power series.

   Solution. This sum rewrites as
$$\frac{\dim_C(W)}{1-\dim_C(V)t} \quad + \sum_{\mathcal{C}\in\mathrm{Conj}(G),\mathcal{C}\neq[\mathrm{Id}]}\frac{\#\mathcal{C}\,\overline{\chi^W(\mathcal{C})}}{1-\chi^V(\mathcal{C})t}.$$
By Q4, the second sum equals $\frac{P(t)}{Q(t)}$, where P and Q are polynomials, and Q is not divisible by $1-\dim_C(V)t$. Then the total sum writes as
$$\frac{\dim_C(W)Q(t)+(1-\dim_C(V)t)P(t)}{(1-\dim_C(V)t)Q(t)}.$$
The nominator is not divisible by $1-\dim_C(V)t$, and hence cannot be the zero polynomial.

6. Conclude.

   Solution. In Q2 we proved the equality of two power series. Further we showed one of them to be zero and the other not. Hence our assumption (the existence of an irrep not appearing as a direct summand of any tensor power of a given faithful rep.) is false.

7. **Example**: Show that for any group G, its left regular representation V^{reg} is faithful. Up to what power n should one go for the assertion we have just shown to hold true?

Solution. Recall that $V^{reg} = \oplus_{g\in G}\mathbb{C}e_g$, and $h \cdot e_g = e_{hg}$. So an $h \in G$ acts trivially on V^{reg} iff $e_{hg} = e_g$ for all $g \in G$, which is possible only for $g = 1$. So, $\rho(h) = \mathrm{Id}$ iff $g = 1$. Thus V^{reg} is a faithful rep. Further, we know that

$$V^{reg} \cong \bigoplus_{V_i \in \mathrm{Irrep}(G)} \dim_C(V_i)V_i.$$

So the 1st power $V^{reg} = (V^{reg})^{\otimes 1}$ already contains all irreps of G as direct summands.

8. **Example**: Show that for any symmetric group S_k, its standard representation V^{st} is faithful. For S_3, up to what power n should one go for the assertion we have just shown to hold true? For S_4, is it sufficient to go up to $n = 2$?

   Solution. Recall that $V^{perm} = \oplus_{1\leq i\leq k}\mathbb{C}e_i$, and $\sigma \cdot e_i = e_{\sigma(i)}$. Here σ acts trivially iff $\sigma(i) = i$ for all i, which means $\sigma = \mathrm{Id}$. Hence V^{perm} is a faithful rep. Now, $V^{perm} \cong V^{st} \oplus V^{tr}$. So, $\rho^{perm}(\sigma) = \mathrm{Id}_k$ iff $\rho^{st}(\sigma) = \mathrm{Id}_{k-1}$ and $\rho^{tr}(\sigma) = \mathrm{Id}_1$. Since $\rho^{tr}(\sigma)$ is always the identity, one concludes that $\rho^{perm}(\sigma) = \mathrm{Id}_k$ iff $\rho^{st}(\sigma) = \mathrm{Id}_{k-1}$. Thus the faithfulness of V^{perm} implies that of V^{st}.
   For S_3, we have seen in class that $V^{\otimes 2} \cong V \oplus V^{\mathbf{sgn}} \oplus V^{tr}$, so $n = 2$ suffices. Here and below $V = V^{st}$.
   For S_4, however, one should go further. One can check using the inner product of characters that, for instance, $V^{\mathbf{sgn}}$ is not contained in $V^{\otimes 2}$. Alternatively, one can use the results from HW3 on the structure of $\Lambda^2(V)$ and $S^2(V)$.