Tutorial 3: Tensor products of representations. Representations of S_n and Young diagrams.

Exercise 1. Our aim is the following isomorphism of S_n-representations, $n \geq 3$:

$$V_{n-1,1} \cong V^{st}.$$

1. For $1 \leq i \leq n$, denote by T_i the Young tableau of shape $(n-1,1)$ having i in the second row, and the remaining numbers from $\{1,\dots,n\}$ arranged in the increasing order in the first row. Show that their classes $[T_i]$ form a complete list of Young tabloids of shape $(n-1,1)$, without repetition.
2. Recall that a permutation $\sigma \in S_n$ acts on a Young tableau by acting on each of its cells. Compute $\sigma \cdot [T_i]$ for any $1 \leq i \leq n$.
3. Identify the above S_n-representation structure on the vector space $\mathbb{C}M_{n-1,1}$ of linear combinations of Young tabloids shape $(n-1,1)$.
4. Determine the column groups C_{T_j} of all the Young tableaux T_j.
5. Express the Young polytabloids $\mathcal{E}_{T_j}$ in terms of the basis $(e_{[T_i]})_{1\leq i\leq n}$ of $\mathbb{C}M_{n-1,1}$.
6. Show that the $\mathcal{E}_{T_j}$ span a sub-representation of $\mathbb{C}M_{n-1,1}$ isomorphic to the standard representation of S_n.

Exercise 2. Representations of S_6

1. Write down all partitions of 6.
2. How many irreps does S_6 have?
3. To what partitions do the irreps V^{tr}, $V^{\mathbf{sgn}}$, V^{st}, and $V^{st} \otimes V^{\mathbf{sgn}}$ correspond?
4. Write down the characters of these four irreps.
5. Determine the degree of the irrep $V_{2,2,2}$ using two methods:
   (a) first, by counting standard Young tableaux;
   (b) then, by the hook length formula.
6. Identify the representation $V_{2,2,2} \otimes V^{\mathbf{sgn}}$.
7. For which irreps V of S_6 does one have an isomorphism of representations $V \cong V \otimes V^{\mathbf{sgn}}$? Give the answer in terms of Specht irreps.
8. Determine the degrees of all irreps of S_6 using your favourite method(s).
9. Check that the sum of the squares of these degrees takes the expected value.
10. Recall the classical symmetric group inclusions $\iota_n\colon S_n \to S_{n+1}$, given by
$$\iota(\sigma)(k) = \begin{cases} \sigma(k), & k \leq n; \\ n+1, & k = n+1. \end{cases}$$
Decompose into irreps the representations $\iota_5^*(V_{2,2,2})$ of S_5, and $\iota_4^*\iota_5^*(V_{2,2,2})$ of S_4.
11. Compute the value of the character χ of $V_{2,2,2}$ on Id, (23), and (23)(45). Show that $\chi((23)(45)(16)) \in \{3,5\}$. (*Hint:* Use the explicit basis for $V_{2,2,2}$ given by the Young polytabloids of the standard Young tableaux of shape $(2,2,2)$.)
12. Using computations from Question 11, give a new proof of the fact that the irreps $V_{2,2,2}$ and $V_{2,2,2} \otimes V^{\mathbf{sgn}}$ are non-isomorphic.
13. What are the degrees of the reps $\Lambda^2(V^{st}), S^2(V^{st}), \Lambda^2(V_{2,2,2}), S^2(V_{2,2,2})$?
14. Compute the value of the characters of $\Lambda^2(V^{st}), S^2(V^{st}), \Lambda^2(V_{2,2,2})$, and $S^2(V_{2,2,2})$, on Id, (23), (23)(45), and (23)(45)(16). (For the last permutation, it suffices to give two possible values.)

15. Deduce from these computations that each of the four reps from the previous point has an irreducible direct summand of degree 9 or 10.
16. How would you check that $\Lambda^2(V^{st})$ is an irrep? (You do not have to carry out the computation. Simply explain what formulas and properties you would use.) From now on, you can assume it.
17. Describe all irreps of degree 10.
18. Check that $S^2(V^{st})$ contains an irreducible direct summand of degree 9. (*Hint:* Use Questions 15 and 17, and evaluate the characters for possible decompositions of $S^2(V^{st})$ on (23)(45), then on (23), then on (123).)
19. Show that the second irrep of degree 9 can be found inside $S^2(V^{st}) \otimes V^{\mathbf{sgn}}$.

Exercise 3. Our aim is to show that, given a faithful representation (V, ρ) of a finite group G, any irrep W of G is contained in some of the tensor powers $V^{\otimes n}$. Recall that

- a representation $\rho\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is called *faithful* if the map ρ is injective;
- the *tensor powers* $V^{\otimes n}$ are defined as $(\ldots((V \otimes V) \otimes V)\ldots) \otimes V$, with n copies of V.

1. Express the character $\chi^{V^{\otimes n}}$ in terms of χ^V.
2. For any representations V and W of a finite group G, prove the following equality of formal power sums:
$$\sum_{n\geq 0}((\chi^V)^n, \chi^W)t^n = \frac{1}{\#G} \sum_{\mathcal{C}\in\mathrm{Conj}(G)} \frac{\#\mathcal{C}\,\overline{\chi^W(\mathcal{C})}}{1-\chi^V(\mathcal{C})t}.$$
Here $\chi(\mathcal{C})$ is defined as $\chi(g)$ for any $g \in \mathcal{C}$.
3. Now suppose V faithful, and W irreducible and not contained in any $V^{\otimes n}$. Show that all coefficients of the formal power series $\sum_{n\geq 0}((\chi^V)^n, \chi^W)t^n$ are then zero.
4. Prove that $\chi^V(\mathcal{C}) = \dim_{\mathbb{C}}(V)$ if and only if $\mathcal{C}$ is the class of 1.
5. Deduce from this that $\sum_{\mathcal{C}\in\mathrm{Conj}(G)} \frac{\#\mathcal{C}\,\overline{\chi^W(\mathcal{C})}}{1-\chi^V(\mathcal{C})t}$ cannot be the zero power series.
6. Conclude.
7. **Example**: Show that for any group G, its left regular representation V^{reg} is faithful. Up to what power n should one go for the assertion we have just shown to hold true?
8. **Example**: Show that for any symmetric group S_k, its standard representation V^{st} is faithful. For S_3, up to what power n should one go for the assertion we have just shown to hold true? For S_4, is it sufficient to go up to $n = 2$?