Tutorial 2: Characters

Exercise 1. Recall that, given a surjective group morphism $\phi\colon G \to H$, there is a map

$$\phi^*\colon \operatorname{Irrep}(H) \to \operatorname{Irrep}(G),$$
$$\rho \mapsto \rho\phi$$

preserving degrees (Tutorial 1). Using characters, prove that ϕ^* is injective.

Solution. Suppose the contrary: non-isomorphic irreps (V_i, ρ_i) of H are sent by ϕ^* to isomorphic irreps $(V_i, \rho_i\phi)$ of G; here $i \in \{1, 2\}$. For characters, this means $\chi^{(V_1,\rho_1\phi)} = \chi^{(V_2,\rho_2\phi)}$. Let us prove that then $\chi^{(V_1,\rho_1)} = \chi^{(V_2,\rho_2)}$, which is impossible since our irreps are non-isomorphic (see Theorem 3).

Take any $h \in H$, and using the surjectivity of ϕ write it as $\phi(g)$ for some $g \in G$. Then $\chi^{(V_1,\rho_1)}(h) = \operatorname{tr}(\rho_1(h)) = \operatorname{tr}(\rho_1\phi(g)) = \chi^{(V_1,\rho_1\phi)}(g) = \chi^{(V_2,\rho_2\phi)}(g) = \cdots = \chi^{(V_2,\rho_2)}(h)$.

Exercise 2. Suppose that the character of a representation V of a finite group G vanishes on all $g \neq 1$. Show that V is then isomorphic to a direct sum $V^{reg} \oplus \cdots \oplus V^{reg}$ of several copies of the regular representation.

Solution. Decompose V into irreps: $V \cong \oplus m_i V_i$, giving $\chi^V = \sum m_i \chi^{V_i}$. Recall the basic properties of V^{reg}: $\chi^{V^{reg}}(g) = \delta_{g,1}\#G$, and $V^{reg} \cong \oplus \dim_{\mathbb{C}}(V_i)V_i$. The first one gives $\chi^V = \frac{\dim_{\mathbb{C}}(V)}{\#G}\chi^{V^{reg}}$. Since the characters of irreps are linearly independent class functions, the second property then yields $m_i = \frac{\dim_{\mathbb{C}}(V)}{\#G}\dim_{\mathbb{C}}(V_i)$ for all i. One of the irreps, say V_1, is the trivial one. Then $\frac{\dim_{\mathbb{C}}(V)}{\#G} = m_1$ is a positive integer. Hence $\chi^V = m_1\chi^{V^{reg}} = \chi^{m_1 V^{reg}}$. The reps V and $m_1 V^{reg}$ have the same character, and are thus isomorphic.

Exercise 3. (Character table for S_4)

1. List all conjugacy classes of S_4. Compute their size.

   Solution. There is one conjugacy class per cycle type (Theorem 7). Further, the number of distinct k-cycles in S_n is $\frac{n(n-1)\cdots(n+1-k)}{k}$ (count the number of possible choices for the first, the second, ... element of your cycle, and divide by k since your cycle can be read starting from any place). Finally, there are 3 elements of the cycle type $(2,2)$ (uniquely defined by where they send, say, 1).

$\#\mathcal{C}$	1	6	8	6	3
$\mathcal{C}$	[Id]	[(12)]	[(123)]	[(1234)]	[(12)(34)]

   Double-checking: $1 + 6 + 8 + 6 + 3 = 24 = \#S_4$.

2. How many (pairwise non-isomorphic) irreducible representations does S_4 have?

   Solution. $\#\operatorname{Irrep}(S_4) = \#\operatorname{Conj}(S_4) = 5$.

3. Determine all 1-dimensional representations.

   Solution. We are looking for group maps $\rho\colon S_4 \to \mathbb{C}^*$. As usual, they coincide with their characters, and thus are class functions. In particular, they take the same value ω on all 2-cycles. $(12)^2 = \mathrm{Id} \implies \omega^2 = 1 \implies \omega = \pm 1$. $(123) = (12)(23) \implies \rho((123)) = \omega^2 = 1$. $(1234) = (123)(34) \implies \rho((1234)) = 1\omega = \omega$. $\rho((12)(34)) = \omega^2 = 1$. There are thus only 2 possibilities for ρ: one with $\omega = 1$, which yields the trivial rep.; and one with $\omega = -1$, which yields the sign rep.: $\rho(\sigma) = \mathbf{sgn}(\sigma)$ is the sign of the permutation σ.

$\#\mathcal{C}$	1	6	8	6	3
V \ $\mathcal{C}$	[Id]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
V^{tr}	1	1	1	1	1
$V^{\mathbf{sgn}}$	1	-1	1	-1	1

4. Find the dimensions of the remaining irreducible representations.

   Solution. $1 + 1 + d_3^2 + d_4^2 + d_5^2 = \#S_4 = 24 \implies d_3^2 + d_4^2 + d_5^2 = 22$, with all $d_i \geq 2$. One cannot have $d_i \geq 4$ for some i, since this would give $d_3^2 + d_4^2 + d_5^2 \geq 4 + 4 + 16 = 24$. So all $d_i \in \{2, 3\}$. By a direct verification, the only possibility (up to re-ordering) is $d_3 = 2, d_4 = d_5 = 3$.

5. Recall the permutation representation $V^{perm} = \oplus_{i=1}^4 \mathbb{C}e_i$, with $\sigma \cdot e_i = e_{\sigma(i)}$. Show that the vector $e_1 + e_2 + e_3 + e_4$ is a basis of a sub-representation L, isomorphic to V^{tr}.

   Solution. For all $\sigma \in S_4$, $\sigma \cdot \sum e_i = \sum e_{\sigma(i)} = \sum e_i$. Moreover, the vector $e_1 + e_2 + e_3 + e_4$ is non-zero.

6. Explain why L admits an S_4-invariant complement. Denote it by (V^{st}, ρ_{st}). It is the *standard representation*, which you saw in Lecture 2 for S_3, and which you will encounter for all groups S_n in the homework.

   Solution. It follows from Maschke's theorem.

7. Use $V^{perm} \cong V^{tr} \oplus V^{st}$ to compute the degree and the character of V^{st}.

   Solution. $\chi^{V^{st}} = \chi^{V^{perm}} - \chi^{V^{tr}}$, $\chi^{V^{perm}}(\sigma)$ is the number of elements of $\{1, 2, 3, 4\}$ fixed by σ. The degree of V^{st} is $\chi^{V^{st}}(\mathrm{Id}) = 3$.

$\#\mathcal{C}$	1	6	8	6	3
V \ $\mathcal{C}$	[Id]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
V^{tr}	1	1	1	1	1
$V^{\mathbf{sgn}}$	1	-1	1	-1	1
V^{st}	3	1	0	-1	-1
$V^{tr} \oplus V^{st} \cong$ V^{perm}	4	2	1	0	0

8. Deduce that V^{st} is irreducible.

   Solution. Use the irreducibility criterion (Theorem 4):
   $$(\chi^{V^{st}}, \chi^{V^{st}}) = \frac{1}{\#S_4} \sum_{\mathcal{C} \in \mathrm{Conj}(S_4)} \#\mathcal{C} |\chi^{V^{st}}(\sigma_{\mathcal{C}})|^2 = \frac{1}{24}(1 * 9 + 6 * 1 + 8 * 0 + 6 * 1 + 3 * 1) = 1.$$
   Here $\sigma_{\mathcal{C}}$ is any permutation from the conjugacy class $\mathcal{C}$.

9. Consider the vector space V^{st} with a different S_4-action: $\rho_{st,\mathbf{sgn}}(\sigma) = \mathbf{sgn}(\sigma)\rho_{st}(\sigma)$. Show that this defines an S_4-representation. Denote it by $V^{st,\mathbf{sgn}}$.

   Solution. $\rho_{st,\mathbf{sgn}}(\sigma\sigma') = \mathbf{sgn}(\sigma\sigma')\rho_{st}(\sigma\sigma') = (\mathbf{sgn}(\sigma)\,\mathbf{sgn}(\sigma'))(\rho_{st}(\sigma)\rho_{st}(\sigma'))$
   $= (\mathbf{sgn}(\sigma)\rho_{st}(\sigma))(\mathbf{sgn}(\sigma')\rho_{st}(\sigma')) = \rho_{st,\mathbf{sgn}}(\sigma)\rho_{st,\mathbf{sgn}}(\sigma')$.

10. Using characters, check that $V^{st,\mathbf{sgn}}$ is an irreducible representation, and $V^{st,\mathbf{sgn}} \ncong V_{st}$.

   Solution. Using the linearity of the trace, one writes $\chi^{V^{st,\mathbf{sgn}}}(\sigma) = \mathrm{tr}(\rho_{st,\mathbf{sgn}}(\sigma)) = \mathrm{tr}(\mathbf{sgn}(\sigma)\rho_{st}(\sigma)) = \mathbf{sgn}(\sigma)\,\mathrm{tr}(\rho_{st}(\sigma)) = \mathbf{sgn}(\sigma)\chi^{V^{st}}(\sigma)$. It differs from $\chi^{V^{st}}$ by signs only, so $(\chi^{V^{st,\mathbf{sgn}}}, \chi^{V^{st,\mathbf{sgn}}})(\chi^{V^{st}}, \chi^{V^{st}}) = 1$. Finally, $\chi^{V^{st,\mathbf{sgn}}} \neq \chi^{V^{st}}$ since they differ on (12).

11. Complete the character table by computing the character of the remaining irreducible representation, denoted by W. (Use the regular representation.)

   Solution.

	$\#\mathcal{C}$	1	6	8	6	3
	$V \backslash \mathcal{C}$	[Id]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
	V^{tr}	1	1	1	1	1
	$V^{\mathbf{sgn}}$	1	-1	1	-1	1
	V^{st}	3	1	0	-1	-1
	$V^{st,\mathbf{sgn}}$	3	-1	0	1	-1
	W	2	0	-1	0	2
$V^{tr} \oplus V^{st} \cong$	V^{perm}	4	2	1	0	0
$V^{tr} \oplus V^{\mathbf{sgn}} \oplus 2W \oplus 3V^{st} \oplus 3V^{st,\mathbf{sgn}} \cong$	V^{reg}	24	0	0	0	0

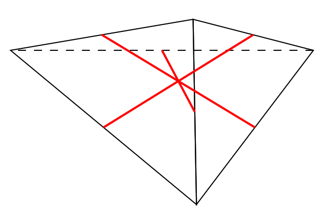
12. Check that the scalar products (χ^W, χ^W) and $(\chi^W, \chi^{V^{st}})$ take the expected values.
13. Check also the orthogonality relations for some of the columns.

In the remainder of the exercise we will describe W explicitly.

14. Recall how to realise S_4 as the group of symmetries of the regular tetrahedron.

   Solution. A symmetry of the regular tetrahedron is entirely determined by how it permutes its 4 vertices. Moreover, any vertex permutation is realised by a symmetry.

15. Verify that this S_4-action permutes the three segments connecting the midpoints of the opposite edges:



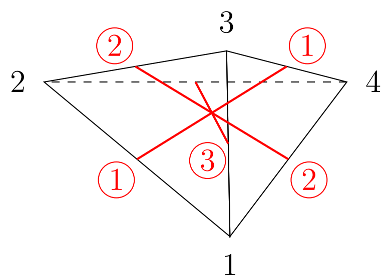
   Solution. A symmetry permutes edges, hence their midpoints. Moreover, opposite edges remain such after a symmetry.

16. Deduce from this a group morphism $\pi\colon S_4 \to S_3$.

   Solution. The effect of a composition of symmetries on the "midpoint segments" is the action of the first symmetry composed with the action of the second one. Hence sending a tetrahedron symmetry to the induced permutation of the "midpoint segments" defines the desired group morphism.

17. For a 2-cycle / a 3-cycle σ of S_4, compute $\pi(\sigma)$.

    Solution. Number the vertices and the midpoints segments as below:



    Now, the permutation (12) fixes the midpoint of the edge 12, and hence the segment ①. Further, it swaps the edges (13) and (23), and hence their midpoints; so it swaps the segments ② and ③. Conclusion: $\pi((12)) = (23)$. Similarly, (123) rotates the triangle 123, giving $\pi((123)) = (123)$.

18. Conclude that π is surjective.

    Solution. Repeating the above argument for different 2- and 3-cocycles of S_4, one sees that all 2- and 3-cocycles of S_3 are in the image of π. Also $\pi(\mathrm{Id}) = \mathrm{Id}$.

19. One then has the injective map $\pi^* \colon \mathrm{Irrep}(S_3) \to \mathrm{Irrep}(S_4)$, $(V, \rho) \mapsto (V, \rho\pi)$ (Exercise 1). Identify the image by π^* of the three irreducible representations of S_3.

    Solution. Comparing the character tables, one sees that π^* send the trivial and the sign irreps of S_3 to the trivial and the sign irreps of S_4 respectively. The standard irrep is sent to W (which is clear already at the level of degrees: W is the only irrep of S_4 of degree 2, and π^* preserves the degrees). Thus W gets a concrete realisation: it is the vector space $\{\sum_{i=1}^{3} \alpha_i e_i | \alpha_i \in \mathbb{C}, \sum \alpha_i = 0\}$, with $\sigma \cdot (\sum \alpha_i e_i) = \sum \alpha_i e_{\pi(\sigma)(i)}$.

Remark: There are other ways to describe W. One can see S_4 as the symmetries of a cube (by looking at its actions on the four diagonals), and send it to S_3 by tracing its action on the three segments connecting the midpoints of the opposite faces. Alternatively, one can consider the action of S_4 on itself by conjugation, and observe that it permutes the three permutations of cycle type $(2, 2)$.

Exercise 4. (Character table for A_4)

Recall that the *alternating group* A_4 is the group of all even permutations in S_4.

1. List all conjugacy classes of A_4. Compute their size.

    Solution. Since A_4 is a sub-group of S_4, its conjugacy classes are sub-classes of those for S_4. The classes of (12) and (1234) disappear, since these permutations are odd. The class of (12)(34) lies entirely in A_4, and becomes a whole conjugacy class of A_4:

$$((123))((12)(34))((123))^{-1} = (23)(14),$$
$$((123))((23)(14))((123))^{-1} = (13)(24).$$

The class of (123) also lies entirely in A_4, but splits into two conjugacy classes of A_4, $[(123)] = \{(123), (134), (432), (421)\}$ and $[(132)] = \{(132), (143), (423), (412)\}$. The fact that the permutations inside each class are conjugated follows from $(bad)(abc)(bad)^{-1} = (dac)$, $(abd)(abc)(abd)^{-1} = (bdc)$, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$. Further, suppose that $\sigma(123)\sigma^{-1} = (132)$ for some $\sigma \in S_4$. Theorem 7 tells that σ should send 1, 2 and 3 either to 1, 3 and 2, or to 3, 2 and 1, or to 2, 1 and 3 (in this order). Then σ is one of (32), (13) and (12), neither of which lie in A_4. Hence $(123) \nsim (132)$ in A_4.

$\#\mathcal{C}$	1	4	4	3
$\mathcal{C}$	[Id]	[(123)]	[(132)]	[(12)(34)]

Double-checking: $1 + 4 + 4 + 3 = 12 = \#A_4$.

2. Find the dimensions of all irreducible representations.

Solution. Taking into consideration the trivial rep., we have $1 + d_2^2 + d_3^2 + d_4^2 = \#A_4 = 12$, so $d_2^2 + d_3^2 + d_4^2 = 11$. If all $d_i \leq 2$, then by parity considerations either 0 or 2 of them equal 2; neither case gives the desired sum of squares. If $d_4 \geq 3$, then $d_2^2 + d_3^2 \leq 2$, so $d_2 = d_3 = 1, d_4 = 3$.

We will now construct the character table for A_4 using three different methods, each of them having a pedagogical interest.

Method 1.

3. Check the following relations in A_4:

$$(123)^3 = \mathrm{Id};$$
$$(123)(134) = (234);$$
$$(123)(124) = (24)(13).$$

4. Using them, determine all 1-dimensional representations of A_4.

Solution. Put $\omega = \rho((123))$, $\omega' = \rho((132))$. The above relations translate as

$$\omega^3 = 1;$$
$$\omega^2 = \omega';$$
$$\rho((24)(13)) = \omega\omega' = \omega^3 = 1.$$

There are 3 possible values for ω: 1, ω_0 and ω_0^2, where $\omega_0 = e^{\frac{2\pi i}{3}}$ is the primitive cubic root of 1. We have shown that ω entirely determines our irrep, and that we should have 3 different irreps of degree 1. Hence all the 3 possibilities for ω are realisable.

5. Using the regular representation, compute the character of the remaining irrep.

Solution.

$\#\mathcal{C}$	1	4	4	3
$V \backslash \mathcal{C}$	[Id]	[(123)]	[(132)]	[(12)(34)]
V^{tr}	1	1	1	1
V_2	1	ω_0	ω_0^2	1
V_3	1	ω_0^2	ω_0	1
V_4	3	0	0	-1
$V^{tr} \oplus V_2 \oplus V_3 \oplus 3V_4 \cong V^{reg}$	12	0	0	0

Method 2.

6. Consider the inclusion map $\iota\colon A_4 \to S_4$. Recall the associated map $\iota^*\colon \mathrm{Rep}(S_4) \to \mathrm{Rep}(A_4)$, $(V,\rho) \mapsto (V,\rho\iota)$. For all irreps of S_4, find the characters of their image by ι^*. What irreps of A_4 are obtained this way?

Solution.

$\#\mathcal{C}$	1	4	4	3
V \ $\mathcal{C}$	[Id]	[(123)]	[(132)]	[(12)(34)]
ι^*V^{tr}	1	1	1	1
$\iota^*V^{\mathbf{sgn}}$	1	1	1	1
ι^*V^{st}	3	0	0	-1
$\iota^*V^{st,\mathbf{sgn}}$	3	0	0	-1
ι^*W	2	-1	-1	2

Computing the inner product of these characters with themselves, one checks that $\iota^*V^{tr} = \iota^*V^{\mathbf{sgn}}$ and $\iota^*V^{st} = \iota^*V^{st,\mathbf{sgn}}$ are irreps, while ι^*W is not. The first one is the trivial irrep.

7. Show that $\iota^*(W)$ decomposes as $L_1 \oplus L_2$, where the sub-representations L_j have degree 1.

Solution. By dimension considerations, this is the only possibility for a reducible rep. of degree 2.

8. Using characters, prove that neither $L_1 \cong L_2$ nor $L_j \cong V^{tr}$ is possible.

Solution. The first case would give $\chi^{L_1} = \frac{1}{2}\chi^{\iota^*(W)}$, and the second one χ^{L_1} or $\chi^{L_2} = \chi^{\iota^*(W)} - \chi^{V^{tr}}$. In any case, some values of these characters would not be roots of 1, which is impossible for degree 1 reps.

9. Conclude that V^{tr}, L_1, L_2 is the complete list of degree 1 irreps of A_4.

Solution. We have seen that these are pairwise non-isomorphic degree 1 irreps, and that A_4 has precisely 3 of them (question 2).

10. Using $\iota^*(W) \cong L_1 \oplus L_2$ and basic properties of characters, finish the character table.

Solution. Let us recall what we have already established:

	$\#\mathcal{C}$	1	4	4	3
	V \ $\mathcal{C}$	[Id]	[(123)]	[(132)]	[(12)(34)]
	V^{tr}	1	1	1	1
	L_1	1	x_1	x_2	x_3
	L_2	1	x_1'	x_2'	x_3'
	$\iota^*V^{st,\mathbf{sgn}}$	3	0	0	-1
$L_1 \oplus L_2 \cong$	ι^*W	2	-1	-1	2

The decomposition $\iota^*(W) \cong L_1 \oplus L_2$ implies $x_1 + x_1' = x_2 + x_2' = -1$ and $x_3 + x_3' = 2$. Further, all the six unknown values are roots of 1. This already yields $x_3 = x_3' = 1$. Now, $(\chi^{V^{tr}}, \chi^{L_1}) = (\chi^{V^{tr}}, \chi^{L_2}) = 0$ translates as $x_1 + x_2 = x_1' + x_2' = -1$. Thus $x_2' = x_1, x_1' = x_2 = -1 - x_1$. Since x_1 and $-1 - x_1$ are roots of 1, one has $1 = (-1-x_1)\overline{(-1-x_1)} = 1 + |x_1|^2 + (x_1 + \overline{x_1}) = 2 + 2\,\mathrm{Re}(x_1) \implies \mathrm{Re}(x_1) = -\frac{1}{2} \implies x_1 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, which are precisely the values ω_0 and ω_0^2 obtained in Method 1. All in all, we recover the same character table using a completely different method.

Method 3.

11. Compose the inclusion map $\iota\colon A_4 \to S_4$ with the group morphism $\pi\colon S_4 \to S_3$ from the previous exercise. Show that the image of $\pi\iota$ is the subgroup $\{\mathrm{Id}, (123), (132)\}$ of S_3, isomorphic to $\mathbb{Z}_3$.

   Solution. Let us consider all types of permutations from A_4. In Exercise 3, we saw that ι send 3-cycles to 3-cycles. Moreover, when acting on the tetrahedron, elements of cycle type $(2,2)$, like $(12)(34)$, reflect some edges around their midpoints, and exchange other edges with their opposites. In any case, each midpoints segment remains in its place. Thus $\iota((12)(34)) = \mathrm{Id}$.

This yields a surjective group morphism $\pi'\colon A_4 \to \mathbb{Z}_3$, and hence an injective map

$$(\pi')^*\colon \mathrm{Irrep}(\mathbb{Z}_3) \to \mathrm{Irrep}(A_4).$$

12. Recall all irreps of $\mathbb{Z}_3$, and describe their images by $(\pi')^*$.

   Solution. Identifying $\mathbb{Z}_3$ with $\{\mathrm{Id}, (123), (132)\}$, its character table reads

$\#\mathcal{C}$	1	1	1
V \ $\mathcal{C}$	$[1]$	$[(123)]$	$[(132)]$
V^{tr}	1	1	1
V_2	1	ω_0	ω_0^2
V_3	1	ω_0^2	ω_0

   By computations from the previous question, $(\pi')^*$ sends the 3 irreps of $\mathbb{Z}_3$ to 3 irreps of A_4, whose characters are obtained by adding on the right a column of 1's (corresponding to the trivial action of $(12)(34)$).

13. Finish the character table.

   Solution. One computes the character of the remaining degree 3 irrep using V^{reg}, and obtains precisely the same table as with the preceding methods.

We will now use the character table to study tensor products of irreps of A_4.

14. Decompose into irreps $V_i \otimes V_j$ for all $V_i, V_j \in \mathrm{Irrep}(A_4)$.

   Solution. We do here only the most difficult case, $V_4 \otimes V_4$. Its character is $\chi^{V_4 \otimes V_4} = \chi^{V_4}\chi^{V_4} = (9, 0, 0, 1)$ (we give its values on the 4 conjugacy classes of A_4, ordered as in our tables). Then $(\chi^{V_4\otimes V_4}, \chi^{V^{tr}}) = \frac{1}{12}(1 \cdot 9 \cdot 1 + 3 \cdot 1 \cdot 1) = 1$, and similarly $(\chi^{V_4\otimes V_4}, \chi^{V_2}) = (\chi^{V_4\otimes V_4}, \chi^{V_3}) = 1$. Then, by a dimension argument, $V_4 \otimes V_4 \cong V^{tr} \oplus V_2 \oplus V_3 \oplus 2V_4$. (One could also compute directly: $(\chi^{V_4\otimes V_4}, \chi^{V_4}) = \frac{1}{12}(1 \cdot 9 \cdot 3 + 3 \cdot 1 \cdot (-1)) = 2$.)

Exercise 5. Show that for a map $\phi\colon G \to \mathbb{C}$, the following two conditions are equivalent:

- $\phi(hgh^{-1}) = \phi(g)$ for all $g, h \in G$;
- $\phi(ab) = \phi(ba)$ for all $a, b \in G$.

Remark: This gives two alternative ways to define class functions.

Solution. In one direction, take $g = ba, h = a$. In the opposite direction, take $a = h, b = gh^{-1}$.