Tutorial 2: Characters

Exercise 1. Recall that, given a surjective group morphism $\phi\colon G \to H$, there is a map

$$\phi^*\colon \operatorname{Irrep}(H) \to \operatorname{Irrep}(G),$$
$$\rho \mapsto \rho\phi$$

preserving degrees (Tutorial 1). Using characters, prove that ϕ^* is injective.

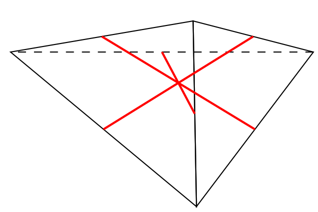
Exercise 2. Suppose that the character of a representation V of a finite group G vanishes on all $g \neq 1$. Show that V is then isomorphic to a direct sum $V^{reg} \oplus \cdots \oplus V^{reg}$ of several copies of the regular representation.

Exercise 3. (Character table for S_4)

1. List all conjugacy classes of S_4. Compute their size.
2. How many irreducible representations does S_4 have?
3. Determine all 1-dimensional representations.
4. Find the dimensions of the remaining irreducible representations.
5. Recall the permutation representation $V^{perm} = \oplus_{i=1}^4 \mathbb{C}e_i$, with $\sigma \cdot e_i = e_{\sigma(i)}$. Show that the vector $e_1 + e_2 + e_3 + e_4$ is a basis of a sub-representation L, isomorphic to V^{tr}.
6. Explain why L admits an S_4-invariant complement. Denote it by (V^{st}, ρ_{st}). It is the *standard representation*, which you saw in Lecture 2 for S_3, and which you will encounter for all groups S_n in the homework.
7. Use $V^{perm} \cong V^{tr} \oplus V^{st}$ to compute the degree and the character of V^{st}.
8. Deduce that V^{st} is irreducible.
9. Consider the vector space V^{st} with a different S_4-action: $\rho_{st,\mathbf{sgn}}(\sigma) = \mathbf{sgn}(\sigma)\rho_{st}(\sigma)$. Show that this defines an S_4-representation. Denote it by $V^{st,\mathbf{sgn}}$.
10. Using characters, check that $V^{st,\mathbf{sgn}}$ is an irreducible representation, and $V^{st,\mathbf{sgn}} \ncong V_{st}$.
11. Complete the character table by computing the character of the remaining irreducible representation, denoted by W. (Use the regular representation.)
12. Check that the scalar products (χ^W, χ^W) and $(\chi^W, \chi^{V^{st}})$ take the expected values.
13. Check also the orthogonality relations for some of the columns.

In the remainder of the exercise we will describe W explicitly.

14. Recall how to realise S_4 as the group of symmetries of the regular tetrahedron.
15. Verify that this S_4-action permutes the three segments connecting the midpoints of the opposite edges:



16. Deduce from this a group morphism $\pi\colon S_4 \to S_3$.
17. For a 2-cycle / a 3-cycle σ of S_4, compute $\pi(\sigma)$.
18. Conclude that π is surjective.
19. One then has the injective map $\pi^*\colon \operatorname{Irrep}(S_3) \to \operatorname{Irrep}(S_4)$, $(V, \rho) \mapsto (V, \rho\pi)$ (Tutorial 1). Identify the image by π^* of the three irreducible representations of S_3.

Remark: There are other ways to describe W. One can see S_4 as the symmetries of a cube (by looking at its actions on the four diagonals), and send it to S_3 by tracing its action on the three segments connecting the midpoints of the opposite faces. Alternatively, one can consider the action of S_4 on itself by conjugation, and observe that it permutes the three permutations of cycle type $(2,2)$.

Exercise 4. (Character table for A_4)

Recall that the *alternating group* A_4 is the group of all even permutations in S_4.

1. List all conjugacy classes of A_4. Compute their size.
2. Find the dimensions of all irreducible representations.

We will now construct the character table for A_4 using three different methods, each of them having a pedagogical interest.

Method 1.

3. Check the following relations in A_4:

$$(123)^3 = \mathrm{Id};$$
$$(123)(134) = (234);$$
$$(123)(124) = (24)(13).$$

4. Using them, determine all 1-dimensional representations of A_4.
5. Using the regular representation, compute the character of the remaining irrep.

Method 2.

6. Consider the inclusion map $\iota\colon A_4 \to S_4$. Recall the associated map $\iota^*\colon \mathrm{Rep}(S_4) \to \mathrm{Rep}(A_4)$, $(V,\rho) \mapsto (V, \rho\iota)$. For all irreps of S_4, find the characters of their image by ι^*. What irreps of A_4 are obtained this way?
7. Show that $\iota^*(W)$ decomposes as $L_1 \oplus L_2$, where the sub-representations L_j have degree 1.
8. Using characters, prove that neither $L_1 \cong L_2$ nor $L_j \cong V^{tr}$ is possible.
9. Conclude that V^{tr}, L_1, L_2 is the complete list of degree 1 irreps of A_4.
10. Using $\iota^*(W) \cong L_1 \oplus L_2$ and basic properties of characters, finish the character table.

Method 3.

11. Compose the inclusion map $\iota\colon A_4 \to S_4$ with the group morphism $\pi\colon S_4 \to S_3$ from the previous exercise. Show that the image of $\pi\iota$ is the subgroup $\{\mathrm{Id}, (123), (132)\}$ of S_3, isomorphic to $\mathbb{Z}_3$.

This yields a surjective group morphism $\pi'\colon A_4 \to \mathbb{Z}_3$, and hence an injective map

$$(\pi')^*\colon \mathrm{Irrep}(\mathbb{Z}_3) \to \mathrm{Irrep}(A_4).$$

12. Recall all irreps of $\mathbb{Z}_3$, and describe their images by $(\pi')^*$.
13. Finish the character table.

We will now use the character table to study tensor products of irreps of A_4.

14. Decompose into irreps $V_i \otimes V_j$ for all $V_i, V_j \in \mathrm{Irrep}(A_4)$.

Exercise 5. Show that for a map $\phi\colon G \to \mathbb{C}$, the following two conditions are equivalent:

- $\phi(hgh^{-1}) = \phi(g)$ for all $g, h \in G$;
- $\phi(ab) = \phi(ba)$ for all $a, b \in G$.

Remark: This gives two alternative ways to define class functions.