Tutorial 1: Basic notions of representation theory

Exercise 1. Let G and H be two groups, and $\phi\colon G \to H$ a map respecting the products: that is, the relation $\phi(gg') = \phi(g)\phi(g')$ holds for all $g, g' \in G$. Show that ϕ then respects the remaining components of the group structure:

1. $\phi(1_G) = 1_H$;
2. for all $g \in G$, $\phi(g^{-1}) = \phi(g)^{-1}$.

Solution. In $\phi(gg') = \phi(g)\phi(g')$, choose

1. $g = g' = 1_G$;
2. $g' = g^{-1}$.

Exercise 2. Show that the two definitions of a representation given in Lecture 1 (one as a group morphism $G \to \mathrm{Aut}_{\mathbb{C}}(V)$, and one as a linear action $G \times V \to V$) are equivalent.

Solution. The equivalence is constructed as follows:

1. for a group morphism $\rho\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$, a linear action on V can be defined by $g \cdot v = \rho(g)(v)$;
2. for a linear action $\cdot$ on V, the maps $\rho(g)\colon v \mapsto g \cdot v$ are linear automorphisms of V, and satisfy $\rho(gg') = \rho(g)\rho(g')$.

Exercise 3. Given a group G, consider the $\mathbb{C}$-vector space $\mathbb{C}G$ with the basis e_g, $g \in G$.

1. Check that the formula $g \cdot e_{g'} = e_{g'g^{-1}}$ defines a G-representation on $\mathbb{C}G$. It is called the *right regular representation* of G.
2. Show that the map $\phi(e_g) = e_{g^{-1}}$ defines an isomorphism between the left and the right regular representations of G.

Solution.

1. $g_1g_2 \cdot e_{g'} = e_{g'(g_1g_2)^{-1}} = e_{g'g_2^{-1}g_1^{-1}} = g_1 \cdot (g_2 \cdot e_{g'})$; $1e_{g'} = e_{g'1} = e_{g'}$.
2. ϕ is a linear map since it is defined on a basis. It is bijective, since $\phi\phi = \mathrm{Id}$. Finally, it is G-linear: $\phi(g' \cdot e_g) = \phi(e_{g'g}) = e_{(g'g)^{-1}} = e_{g^{-1}g'^{-1}} = g' \cdot \phi(e_g)$. Here the first and the last actions $\cdot$ should be understood in the sense of the left and the right regular representations of G respectively.

Exercise 4. Consider a group morphism $\phi\colon G \to H$.

1. Check that for any representation $\rho\colon H \to \mathrm{Aut}_{\mathbb{C}}(V)$ of H, the map $\rho\phi$ yields a representation of G.
2. Show that this gives a monoid morphism
$$\phi^*\colon \mathrm{Rep}(H) \to \mathrm{Rep}(G),$$
$$\rho \mapsto \rho\phi,$$
where the monoid structures are those described in Lecture 3.
3. Assuming ϕ surjective, show that ϕ^* restricts to a map $\mathrm{Irrep}(H) \to \mathrm{Irrep}(G)$.

Remark: This result can be used for constructing irreducible representations of a group out of those of smaller ones. We will see it later for the symmetric groups S_4 and S_3.

Solution.

1. $\rho\phi$ is a group morphism since both ρ and ϕ are so.

2. On the level of vector spaces, ϕ^* is the identity map. So is sends the zero representation of H to the zero representation of G. Similarly, for $(V_i, \rho_i) \in \mathrm{Rep}(H)$, $\phi^*(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ is the vector space $V_1 \oplus V_2$ with the G-representation structure $(\rho_1 \oplus \rho_2)\phi = (\rho_1\phi) \oplus (\rho_2\phi)$, which is precisely $\phi^*(V_1, \rho_1) \oplus \phi^*(V_2, \rho_2)$. (Reminder: the direct sum of linear maps is defined by $(\phi \oplus \psi)(v, w) = (\phi(v), \psi(w))$. The map $\rho_1 \oplus \rho_2 \colon G \to \mathrm{Aut}_{\mathbb{C}}(V_1) \oplus \mathrm{Aut}_{\mathbb{C}}(V_2)$ sends g to $\rho_1(g) \oplus \rho_2(g)$.)
3. Take $(V, \rho) \in \mathrm{Irrep}(H)$, and assume that $(V, \rho\phi)$ is reducible. It then has a G-invariant subspace V' different from $\{0\}$ and V itself. We will show that V' is also H-invariant, which is a contradiction. Take $h \in H$. Since ϕ is surjective, $h = \phi(g)$ for some $g \in G$. But then $\rho(h) = \rho(\phi(g)) = (\rho\phi)(g)$, which restricts to V' because V' is G-invariant.

Exercise 5. Consider a finite field $\mathbb{F}_p$, a finite group G whose order is divisible by p, and the left regular representation $\mathbb{F}_pG$ of G over $\mathbb{F}_p$. Define a linear map $\varepsilon \colon \mathbb{F}_pG \to \mathbb{F}_p$ by $\varepsilon(e_g) = 1$ for all $g \in G$. Put $I = \mathrm{Ker}\,\varepsilon$. The aim of the exercise is to show that I is a sub-representation of $\mathbb{F}_pG$ admitting no G-invariant complement.

1. Check that I is a sub-representation of $\mathbb{F}_pG$, which is proper ($\neq \mathbb{F}_pG$) and non-zero. What is its dimension over $\mathbb{F}_p$?
2. Suppose that there is a decomposition $\mathbb{F}_pG = I \oplus V$ of G-representations. Take any non-zero $v \in V$, and put $w = \sum_{g \in G} g \cdot v$. Show that w rewrites as $\sum_{g \in G} \varepsilon(v) e_g$.
3. Deduce from this $w \in I \cap V$.
4. Verify that w is non-zero.
5. Conclude.

Remark: This example shows that over fields of positive characteristic, the complete reducibility for representations of finite groups we have established over $\mathbb{C}$ does not always hold.

Solution.

1. I can be described as $I = \{\sum_{g \in G} \alpha_g e_g | \alpha_g \in \mathbb{F}_p, \sum_{g \in G} \alpha_g = 0\}$. It is G-invariant since the G-action does not change the sum of coefficients. It is proper since $e_1 \notin I$ (as $\varepsilon(e_1) = 1$), and non-zero since $0 \neq e_g - e_1 \in I$ for all $g \neq 1$ (such g exist since $\#G$ is at least p, which is ≥ 2). Writing I as $I = \{\sum_{g \in G, g \neq 1} \alpha_g e_g - (\sum_{g \in G, g \neq 1} \alpha_g) e_1 | \alpha_g \in \mathbb{F}_p, g \in G \setminus \{1\}\}$, one gets $\dim_{\mathbb{F}_p} I = \#G - 1$.
   Alternatively, one can use Lemma 10 from Lecture 8.
2. Write v in the basis e_g: $v = \sum_{h \in G} \alpha_h e_h$. Then $w = \sum_{g \in G} g \cdot v = \sum_{g \in G, h \in G} \alpha_h e_{gh} = \sum_{k \in G, h \in G} \alpha_h e_k = \sum_{k \in G} (\sum_{h \in G} \alpha_h) e_k = \sum_{k \in G} \varepsilon(v) e_k$. We used the change of variables $g \rightsquigarrow k = gh$.
3. $w \in V$ since $w = \sum_{g \in G} g \cdot v$, $v \in V$, and V is G-invariant. $w \in I$ since $\varepsilon(w) = \varepsilon(\sum_{g \in G} \varepsilon(v) e_g) = \varepsilon(v)\varepsilon(\sum_{g \in G} e_g) = \varepsilon(v)\#G = 0$. We used $\#G = 0$ in $\mathbb{F}_p$.
4. We should show that the coefficients of w in the basis e_g, which are all equal to $\varepsilon(v)$, are non-zero. Indeed, $\varepsilon(v) = 0$ would mean $v \in I$. But v was chosen as a non-zero element of V. This would contradict our assumption $V \cap I = \{0\}$ (which is part of the statement $\mathbb{F}_pG = I \oplus V$).
5. Hence our assumption (there is a decomposition $\mathbb{F}_pG = I \oplus V$ of G-representations) was wrong. Thus I is a proper non-zero sub-representation of $\mathbb{F}_pG$ admitting no G-invariant complement.