Tutorial 1: Basic notions of representation theory

Exercise 1. Let G and H be two groups, and $\phi\colon G \to H$ a map respecting the products: that is, the relation $\phi(gg') = \phi(g)\phi(g')$ holds for all $g, g' \in G$. Show that ϕ then respects the remaining components of the group structure:

1. $\phi(1_G) = 1_H$;
2. for all $g \in G$, $\phi(g^{-1}) = \phi(g)^{-1}$.

Exercise 2. Show that the two definitions of a representation given in Lecture 1 (one as a group morphism $G \to \mathrm{Aut}_{\mathbb{C}}(V)$, and one as a linear action $G \times V \to V$) are equivalent.

Exercise 3. Given a group G, consider the $\mathbb{C}$-vector space $\mathbb{C}G$ with the basis e_g, $g \in G$.

1. Check that the formula $g \cdot e_{g'} = e_{g'g^{-1}}$ defines a G-representation on $\mathbb{C}G$. It is called the *right regular representation* of G.
2. Show that the map $\phi(e_g) = e_{g^{-1}}$ defines an isomorphism between the left and the right regular representations of G.

Exercise 4. Consider a group morphism $\phi\colon G \to H$.

1. Check that for any representation $\rho\colon H \to \mathrm{Aut}_{\mathbb{C}}(V)$ of H, the map $\rho\phi$ yields a representation of G.
2. Show that this gives a monoid morphism
$$\phi^*\colon \mathrm{Rep}(H) \to \mathrm{Rep}(G),$$
$$\rho \mapsto \rho\phi,$$
where the monoid structures are those described in Lecture 3.
3. Assuming ϕ surjective, show that ϕ^* restricts to a map $\mathrm{Irrep}(H) \to \mathrm{Irrep}(G)$.

Remark: This result can be used for constructing irreducible representations of a group out of those of smaller ones. We will see it later for the symmetric groups S_4 and S_3.

Exercise 5. Consider a finite field $\mathbb{F}_p$, a finite group G whose order is divisible by p, and the left regular representation $\mathbb{F}_pG$ of G over $\mathbb{F}_p$. Define a linear map $\varepsilon\colon \mathbb{F}_pG \to \mathbb{F}_p$ by $\varepsilon(g) = 1$ for all $g \in G$. Put $I = \mathrm{Ker}\,\varepsilon$. The aim of the exercise is to show that I is a sub-representation of $\mathbb{F}_pG$ admitting no G-invariant complement.

1. Check that I is a sub-representation of $\mathbb{F}_pG$, which is proper ($\neq \mathbb{F}_pG$) and non-zero. What is its dimension over $\mathbb{F}_p$?
2. Suppose that there is a decomposition $\mathbb{F}_pG = I \oplus V$ of G-representations. Take any non-zero $v \in V$, and put $w = \sum_{g \in G} g \cdot v$. Show that w rewrites as $\sum_{g \in G} \varepsilon(v) e_g$.
3. Deduce from this $w \in I \cap V$.
4. Verify that w is non-zero.
5. Conclude.

Remark: This example shows that over fields of positive characteristic, the complete reducibility for representations of finite groups we have established over $\mathbb{C}$ does not always hold.