The remarkable limit

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \]
Trigonometric functions revisited

Trigonometric functions like $\sin(x)$ and $\cos(x)$ are continuous everywhere. Informally, this can be explained as follows: a small perturbation of a point on the unit circle results in small changes in its $x$- and $y$-coordinates.

These functions are periodic, and so have an oscillating behaviour at infinity. Therefore, they have neither a finite nor an infinite limit at infinity.
Trigonometric functions revisited

Example. The limit \( \lim_{x \to 1} \cos \left( \frac{x^2-1}{x-1} \right) \) can be evaluated using what we know about the composition of continuous functions. Indeed, since \( \cos \) is continuous on \( \mathbb{R} \), we have

\[
\lim_{x \to 1} \cos(g(x)) = \cos \lim_{x \to 1} g(x)
\]

whenever \( \lim_{x \to 1} g(x) \) exists. Here \( g(x) = \frac{x^2 - 1}{x - 1} = x + 1 \) for \( x \neq 1 \). Therefore,

\[
\lim_{x \to 1} \cos \left( \frac{x^2 - 1}{x - 1} \right) = \lim_{x \to 1} \cos (x + 1) = \cos \lim_{x \to 1} (x + 1) = \cos 2.
\]
The Squeeze Theorem

Interesting things start to happen when we mix trigonometric and polynomial functions. For instance, one of the most important limits for applications of calculus is $\lim_{x \to 0} \frac{\sin x}{x}$. So far we have not proved any results that would allow to approach this limit.

**Theorem.** $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

**Informal proof.** The key idea of the proof is very simple but very important. Suppose that we have three functions $f(x)$, $g(x)$, and $h(x)$, and that we can prove that:

1. the inequalities $g(x) \leq f(x) \leq h(x)$ hold for all $x$ in some open interval containing the number $c$, possibly excluding $c$ itself;
2. $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$.

Then $\lim_{x \to c} f(x) = L$ as well. This is The Squeeze Theorem: the values of $f$ are “squeezed” between values of $g$ and $h$. It is also called The Sandwich Theorem, or, in some languages, The Two Policemen (and a Drunk) Theorem.
The Squeeze Theorem: illustration
Proving $\lim_{x \to 0} \frac{\sin x}{x} = 1$

We shall apply the Squeeze Theorem for

$$g(x) = \cos x, \quad f(x) = \frac{\sin x}{x}, \quad h(x) = 1 \quad \text{on } (-\pi/2, \pi/2).$$

Why $\cos x \leq \frac{\sin x}{x} \leq 1$?

It is enough to prove it for $(0, \pi/2)$ since the functions involved are even. On that interval, it is the same as $\sin x \leq x \leq \tan x$. 
Proving \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

Why \( \sin x \leq x \leq \tan x \)?

This is proved using the geometric picture

where we can actually find all the quantities involved!
Proving \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

Indeed,

\[
\text{the area of the small triangle is } \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin x = \frac{1}{2} \sin x;
\]
Proving \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

The area of the sector is \( \pi \cdot \frac{x}{2\pi} = \frac{1}{2}x \), since it is the \( \frac{x}{2\pi} \)-th part of the whole disk whose area is \( \pi \cdot 1^2 = \pi \);
Proving $\lim_{x \to 0} \frac{\sin x}{x} = 1$

and the area of the large triangle is $\frac{1}{2} \cdot 1 \cdot \tan x = \frac{1}{2} \tan x$. 
Proving \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

Therefore an obvious inequality between the areas implies

\[ \sin x \leq x \leq \tan x, \]

which is what we needed.
Proving \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

Finally,

\[
\sin x \leq x \leq \tan x
\]

guarantees that for \( 0 < x < \pi/2 \) we have

\[
\cos x \leq \frac{\sin x}{x} \leq 1,
\]

so since our functions are even, we conclude that for \( x \neq 0 \) in \((-\pi/2, \pi/2)\) we have

\[
\cos x \leq \frac{\sin x}{x} \leq 1.
\]

Since \( \lim_{x \to 0} \cos x = \lim_{x \to 0} 1 = 1 \), by the Squeeze Theorem we conclude that

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]
**Consequences of** $\lim_{x \to 0} \frac{\sin x}{x} = 1$

\[
\lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \left[ \frac{\sin x}{x} \cdot \frac{1}{x} \right] = +\infty.
\]

\[
\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left[ \frac{\sin x}{x} \cdot \frac{1}{\cos x} \right] = 1 \cdot 1 = 1.
\]

\[
\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} 2 \frac{\sin 2x}{2x} = \lim_{x \to 0} \frac{2\sin t}{t} = 2.
\]

\[
\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{1-(\cos x)^2}{(1+\cos x)x^2} = \lim_{x \to 0} \frac{(\sin x)^2}{(1+\cos x)x^2} = \lim_{x \to 0} \left[ \left( \frac{\sin x}{x} \right)^2 \frac{1}{1+\cos x} \right],
\]

so \[
\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^2 \lim_{x \to 0} \frac{1}{1+\cos x} = 1^2 \cdot \frac{1}{2} = \frac{1}{2}.
\]

\[
\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \left[ \frac{1-\cos x}{x^2} \cdot x \right] = \frac{1}{2} \cdot 0 = 0.
\]

\[
\lim_{x \to 0} \frac{2-\cos 3x - \cos 4x}{x} = \lim_{x \to 0} \left[ 3 \frac{1-\cos 3x}{3x} + 4 \frac{1-\cos 4x}{4x} \right] = 0 + 0 = 0.
\]
Informal consequences of $\lim_{x \to 0} \frac{\sin x}{x} = 1$

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ means informally that for small $x$ we have $\sin x \approx x$.

2. $\lim_{x \to 0} \frac{\tan x}{x} = 1$ means informally that for small $x$ we also have $\tan x \approx x$.

3. $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ means informally that for small $x$ we have $\cos x \approx 1 - \frac{x^2}{2}$.

These approximate formulas give examples of a general strategy of differential calculus: replacing a function by a polynomial expression that approximates it very well for small $x$ (or $x$ close to the given point $a$).