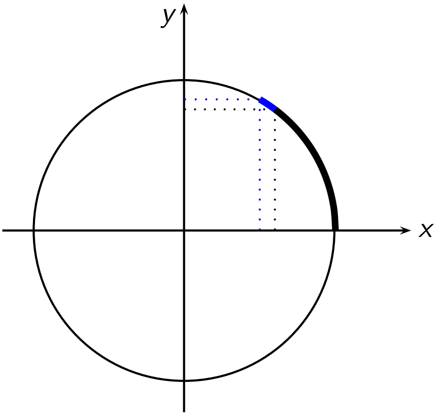
The remarkable limit

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Trigonometric functions revisited

Trigonometric functions like $\sin(x)$ and $\cos(x)$ are continuous everywhere. Informally, this can be explained as follows: a small perturbation of a point on the unit circle results in small changes in its x- and y-coordinates.



These functions are periodic, and so have an oscillating behaviour at infinity. Therefore, they have neither a finite nor an infinite limit at infinity.

Trigonometric functions revisited

Example. The limit $\lim\limits_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right)$ can be evaluated using what we know about the composition of continuous functions. Indeed, since cos is continuous on $\mathbb{R}$, we have

$$\lim_{x\to 1} \cos(g(x)) = \cos \lim_{x\to 1} g(x)$$

whenever $\lim\limits_{x\to 1} g(x)$ exists. Here $g(x) = \dfrac{x^2-1}{x-1} = x+1$ for $x \neq 1$. Therefore,

$$\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right) = \lim_{x\to 1} \cos(x+1) = \cos \lim_{x\to 1}(x+1) = \cos 2.$$

The Squeeze Theorem

Interesting things start to happen when me mix trigonometric and polynomial functions. For instance, one of the most important limit for applications of calculus is $\lim_{x\to 0} \frac{\sin x}{x}$. So far we have not proved any results that would allow to approach this limit.

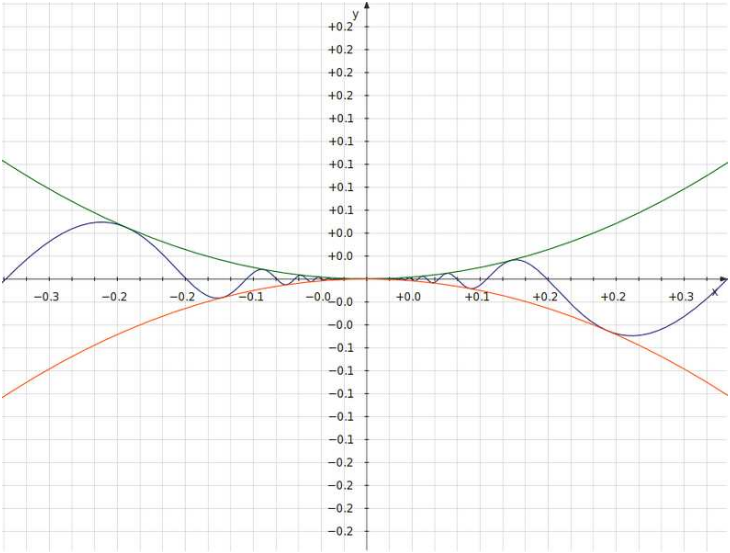
Theorem. $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

Informal proof. The key idea of the proof is very simple but very important. Suppose that we have three functions $f(x)$, $g(x)$, and $h(x)$, and that we can prove that:

1. the inequalities $g(x) \leq f(x) \leq h(x)$ hold for all x in some open interval containing the number c, possibly excluding c itself;
2. $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$.

Then $\lim_{x\to c} f(x) = L$ as well. This is The Squeeze Theorem: the values of f are "squeezed" between values of g and h. It is also called The Sandwich Theorem, or, in some languages, The Two Policemen (and a Drunk) Theorem.

The Squeeze Theorem: illustration



PROVING $\lim_{x \to 0} \frac{\sin x}{x} = 1$

We shall apply the Squeeze Theorem for

$$g(x) = \cos x, \qquad f(x) = \frac{\sin x}{x}, \qquad h(x) = 1 \qquad \text{on } (-\pi/2, \pi/2).$$

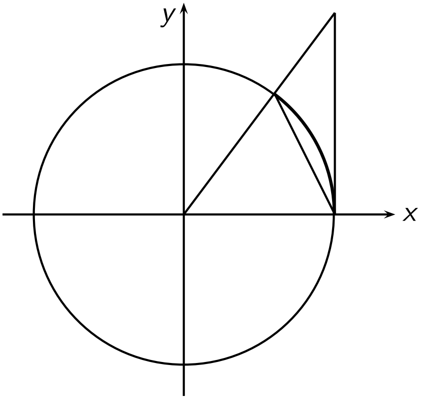
Why $\cos x \leq \frac{\sin x}{x} \leq 1$?

It is enough to prove it for $(0, \pi/2)$ since the functions involved are even. On that interval, it is the same as $\sin x \leq x \leq \tan x$.

PROVING $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Why $\sin x \leq x \leq \tan x$?

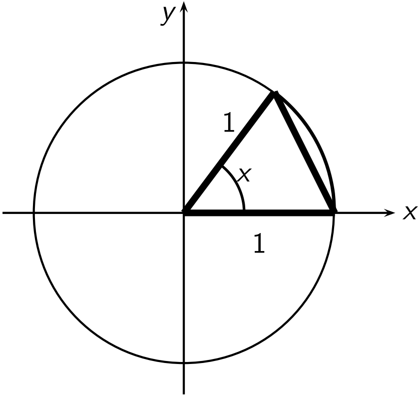
This is proved using the geometric picture



where we can actually find all the quantities involved!

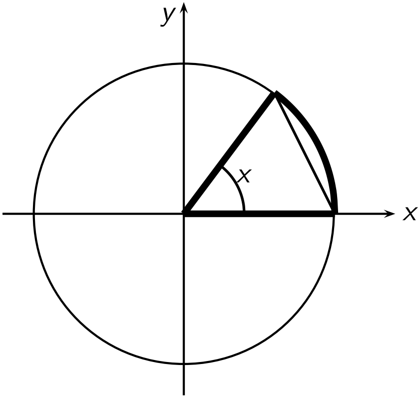
Proving $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Indeed,



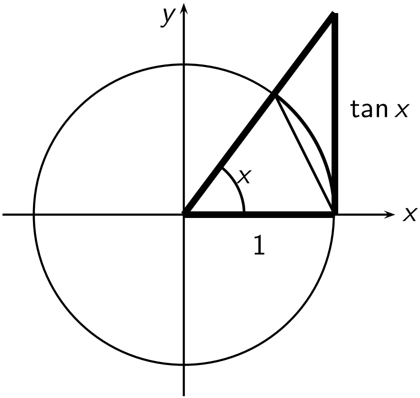
the area of the small triangle is $\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin x = \frac{1}{2} \sin x$;

Proving $\lim_{x \to 0} \frac{\sin x}{x} = 1$



the area of the sector is $\pi \cdot \frac{x}{2\pi} = \frac{1}{2}x$, since it is the $\frac{x}{2\pi}$-th part of the whole disk whose area is $\pi \cdot 1^2 = \pi$;

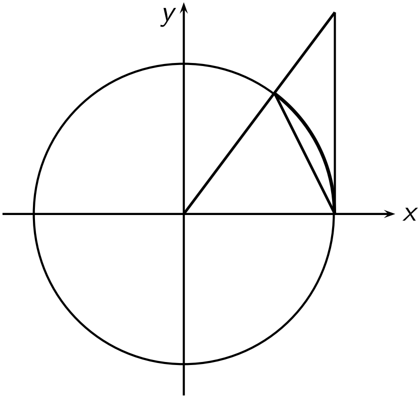
PROVING $\lim_{x \to 0} \frac{\sin x}{x} = 1$



and the area of the large triangle is $\frac{1}{2} \cdot 1 \cdot \tan x = \frac{1}{2} \tan x$.

Proving $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Therefore an obvious inequality between the areas



implies

$$\sin x \le x \le \tan x,$$

which is what we needed.

PROVING $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Finally,

$$\sin x \le x \le \tan x$$

guarantees that for $0 < x < \pi/2$ we have

$$\cos x \le \frac{\sin x}{x} \le 1,$$

so since our functions are even, we conclude that for $x \neq 0$ in $(-\pi/2, \pi/2)$ we have

$$\cos x \le \frac{\sin x}{x} \le 1.$$

Since $\lim_{x \to 0} \cos x = \lim_{x \to 0} 1 = 1$, by the Squeeze Theorem we conclude that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Consequences of $\lim_{x\to 0}\frac{\sin x}{x}=1$

$$\lim_{x\to 0^+}\frac{\sin x}{x^2}=\lim_{x\to 0^+}\left[\frac{\sin x}{x}\cdot\frac{1}{x}\right]=+\infty.$$

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right]=1\cdot 1=1.$$

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}2\frac{\sin 2x}{2x}=\lim_{2x=t\to 0}2\frac{\sin t}{t}=2.$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1-(\cos x)^2}{(1+\cos x)x^2}=\lim_{x\to 0}\frac{(\sin x)^2}{(1+\cos x)x^2}=\lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)^2\frac{1}{1+\cos x}\right],$$
$$\text{so }\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\lim_{x\to 0}\frac{1}{1+\cos x}=1^2\cdot\frac{1}{2}=\frac{1}{2}.$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\left[\frac{1-\cos x}{x^2}\cdot x\right]=\frac{1}{2}\cdot 0=0.$$

$$\lim_{x\to 0}\frac{2-\cos 3x-\cos 4x}{x}=\lim_{x\to 0}\left[3\frac{1-\cos 3x}{3x}+4\frac{1-\cos 4x}{4x}\right]=0+0=0.$$

Informal consequences of $\lim_{x \to 0} \frac{\sin x}{x} = 1$

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ means informally that for small x we have $\sin x \approx x$,
2. $\lim_{x \to 0} \frac{\tan x}{x} = 1$ means informally that for small x we also have $\tan x \approx x$,
3. $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$ means informally that for small x we have $\cos x \approx 1 - \frac{x^2}{2}$.

These approximate formulas give examples of a general strategy of differential calculus: replacing a function by a polynomial expression that approximates it very well for small x (or x close to the given point a).