Quiz 2: Character tables

Instructions. Give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those. The question marked with $*$ is a bonus question.

Exercise 1. Let D_{10} be the group of symmetries of a regular pentagon P. Denote by r and by s respectively a $\frac{2\pi}{5}$-rotation and a reflection, as shown in the figure:

You can assume without proof that the following relations hold in D_{10}:

$$r^5 = s^2 = \mathrm{Id}, \qquad srs = r^4,$$

and that the following 10 symmetries form a complete list of the elements of D_{10}:

$$\mathrm{Id}, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4.$$

Remark. Similarly to what we did in HW2 for the dihedral group D_8, one can construct one of the two degree 2 irreps of D_{10} by extending its action on the vertices of P to an action on $\mathbb{R}^2$, and then on $\mathbb{C}^2$. However, I preferred showing you another method, based on Schur's orthogonality relations for a character table, in order to give you a better idea of an approach we have not seen much in practice.

1. Find an abelian subgroup of D_{10} of index 2.

   Solution. Since $r^5 = \mathrm{Id}$, the elements $\mathrm{Id}, r, r^2, r^3, r^4$ form a subgroup of D_{10}. More precisely, it is a cyclic subgroup of size 5, so it is abelian and has index $\frac{10}{5} = 2$ in D_{10}.

2. Is D_{10} abelian?

   Solution. No: $srs^{-1} = srs = r^4 \neq r$.

3. What do the previous points tell you about the irreps of D_{10}?

   Solution. First, D_{10} has an abelian subgroup of index 2, thus (by a theorem seen in class) the degree of any of its irreps is ≤ 2. Second, D_{10} itself is not abelian, so (by another theorem seen in class) it has at least one irrep of degree 2.

4. Describe all degree 1 representations of D_{10}.

   Solution. A degree 1 representation $\rho\colon D_{10} \to \mathbb{C}^*$ should satisfy $\rho(r)^5 = \rho(s)^2 = 1$, and $\rho(s)\rho(r)\rho(s) = \rho(r)^4$. The last condition implies

   $$\rho(r)^4 = \rho(s)\rho(r)\rho(s) = \rho(s)^2\rho(r) = \rho(r),$$

   so $\rho(r)^3 = 1$, and $\rho(r) = \rho(r)^3\rho(r)^3\rho(r)^{-5} = 1$. So there are only two possibilities for ρ:

(a) $\rho(g) = 1$ for all $g \in D_{10}$, which gives the trivial rep. V^{tr};

(b) $\rho(g) = 1$ for orientation preserving symmetries g, and $\rho(g) = -1$ for orientation reversing g; this is a group morphism, since a product of pentagon symmetries preserves the orientation if and only if either both, or neither of the symmetries do.

5. Determine the number and the degrees of all irreps of D_{10}.

Solution. By Q3, the k remaining irreps of D_{10} are of degree 2. Thus
$$10 = \#D_{10} = 1^2 + 1^2 + k \cdot 2^2 \qquad \Longrightarrow \qquad k = 2.$$
Summarising, D_{10} has two degree 1 irreps, and two degree 2 irreps.

6. Verify that D_{10} has exactly 4 conjugacy classes:
$$[\mathrm{Id}],\ [r],\ [r^2],\ [s].$$

Solution. Our group splits into 4 conjugacy classes, since $\#\operatorname{Conj}(D_{10}) = \#\operatorname{Irrep}(D_{10}) = 4$. Further, $srs^{-1} = r^4$ and $sr^2s^{-1} = (srs^{-1})^2 = r^8 = r^3$, so $r \sim r^4$ and $r^2 \sim r^3$. For all $\alpha \in \mathbb{Z}$, one has $rsr^{\alpha}r^{-1} = sr^4r^{\alpha}r^{-1} = sr^{\alpha+3}$. Since 3 and 5 are coprime, this means $sr^{\alpha} \sim s$ for all α. Thus every element of D_{10} is conjugate to one of Id, r, r^2, s. Since there are 4 classes in total, these 4 elements are pairwise non-conjugate.

In what follows, V_1 denotes the non-trivial degree 1 irrep, and V_2 and V_3 the two degree 2 irreps. You can write $\chi_i := \chi^{V_i}$ for simplicity.
First, suppose that
$$V_2 \otimes V_1 \cong V_3.$$

7. What does it mean for the character table?

Solution. On the level of characters, $V_2 \otimes V_1 \cong V_3$ translates as $\chi_2(g)\chi_1(g) = \chi_3(g)$ for all $g \in D_{10}$. Since $\chi_1(g)$ is -1 for $g \sim s$ and 1 for other classes, this means that the character table should look as follows:

$\#\mathcal{C}$	1	2	2	5
V \ $\mathcal{C}$	$[\mathrm{Id}]$	$[r]$	$[r^2]$	$[s]$
V^{tr}	1	1	1	1
V_1	1	1	1	-1
V_2	2	a	b	c
V_3	2	a	b	$-c$
V^{reg}	10	0	0	0

8. Using the regular representation of D_{10}, complete the character table as far as you can.

Solution. Recall that
$$V^{reg} \cong \oplus_i \dim_{\mathbb{C}}(V_i)V_i = V^{tr} \oplus V_1 \oplus 2V_2 \oplus 2V_3.$$
For the first and the last column, this gives nothing new. For the second one, it yields
$$1 \cdot 1 + 1 \cdot 1 + 2 \cdot a + 2 \cdot a = 0,$$
so $a = -\frac{1}{2}$. Similarly, for the third column it gives $b = -\frac{1}{2}$.

9. Looking at the columns of your table, get a contradiction.

   Solution. The second and the third columns are then non-zero and equal. But, according to Schur's orthogonality relations, they should be orthogonal. Conclusion: our assumption $V_2 \otimes V_1 \cong V_3$ cannot hold.

We are left with the second possibility:

$$V_2 \otimes V_1 \cong V_2, \qquad V_3 \otimes V_1 \cong V_3.$$

Put $x = \chi_2(r), y = \chi_2(r^2)$.

10. Using the regular representation of D_{10}, express all entries of the character table in terms of x and y.

    Solution. From $\chi_2(g)\chi_1(g) = \chi_2(g)$ and $\chi_3(g)\chi_1(g) = \chi_3(g)$ for all $g \in D_{10}$ follows $\chi_2(s) = \chi_3(s) = 0$. For the second column, V^{reg} gives

    $$1 \cdot 1 + 1 \cdot 1 + 2 \cdot x + 2 \cdot \chi_3(r) = 0,$$

    so $\chi_3(r) = -1 - x$. Similarly, for the third column it gives $\chi_3(r^2) = -1 - y$.

$\#\mathcal{C}$	1	2	2	5
$V \backslash \mathcal{C}$	[Id]	$[r]$	$[r^2]$	$[s]$
V^{tr}	1	1	1	1
V_1	1	1	1	-1
V_2	2	x	y	0
V_3	2	$-1-x$	$-1-y$	0
V^{reg}	10	0	0	0

11. From the orthogonality relation for certain rows, deduce

    $$x + y = -1.$$

    (Try to pick the easiest pair of rows!)

    Solution. For the rows of V_2 and V_1, the orthogonality reads

    $$1 \cdot 2 \cdot \overline{1} + 2 \cdot x \cdot \overline{1} + 2 \cdot y \cdot \overline{1} + 5 \cdot 0 \cdot \overline{-1} = 0,$$

    so $x + y = -1$.

12. Express the character of the alternating square $\Lambda^2(V_2)$ in terms of x and y.

    Solution. For the conjugacy class representatives Id, r, r^2, s, we need to determine in what classes fall their squares: $\mathrm{Id}^2 = s^2 = \mathrm{Id}$, $(r)^2 = r^2$, $(r^2)^2 = r^4 \sim r$. Now, from the formula

    $$\chi^{\Lambda^2(V_2)}(g) = \frac{1}{2}(\chi_2(g)^2 - \chi_2(g^2)),$$

    we deduce the character of $\Lambda^2(V_2)$:

$\mathcal{C}^2$	[Id]	$[r^2]$	$[r]$	[Id]
$V \backslash \mathcal{C}$	[Id]	$[r]$	$[r^2]$	$[s]$
V_2	2	x	y	0
$\Lambda^2(V_2)$	1	$\frac{1}{2}(x^2 - y)$	$\frac{1}{2}(y^2 - x)$	-1

13. Show that $\Lambda^2(V_2) \cong V_1$. Deduce from this
$$x^2 - y = 2.$$

Solution. We know that $\Lambda^2(V_2)$ has degree 1, and its character takes the value -1 on s. Out of the two degree 1 irreps of D_{10}, only V_1 satisfies this property. So $\Lambda^2(V_2) \cong V_1$, and their characters coincide. This gives $\frac{1}{2}(x^2 - y) = 1$, hence $x^2 - y = 2$.

14. Using Q11 and Q13, determine the values of x and y.

Solution. From $x + y = -1$ follows $y = -1 - x$, so $x^2 - y = 2$ becomes $x^2 + x - 1 = 0$. Its solutions are $x = \frac{-1\pm\sqrt{5}}{2}$, leading to $y = \frac{-1\mp\sqrt{5}}{2}$. We choose $x = \frac{-1+\sqrt{5}}{2}$, $y = \frac{-1-\sqrt{5}}{2}$; the alternative choice corresponds to interchanging the irreps V_2 and V_3.

15. Complete the character table of D_{10}.

Solution.

D_{10}	[Id]	$[r]$	$[r^2]$	$[s]$
V^{tr}	1	1	1	1
V_1	1	1	1	-1
V_2	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0
V_3	2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0

16. The symmetries of our pentagon P act on its vertices. This yields an injective group morphism $\iota\colon D_{10} \to S_5$ into the symmetric group S_5. Evaluate ι explicitly, without justification, on Id, r, r^2, and s.

Solution. $\iota(\mathrm{Id}) = \mathrm{Id}$, $\iota(r) = (12345)$, $\iota(r^2) = (13524)$, $\iota(s) = (15)(24)$.

17. Using the character table of S_5 recalled below, decompose into irreps the restricted representations $\iota^*(V^{st})$ of D_{10}, where $V^{st} \in \mathrm{Irrep}(S_5)$.

Solution. We first compute the characters of all restricted representations $\iota^*(V)$ of D_{10}, where $V \in \mathrm{Irrep}(S_5)$. (Note that for this question, the row for V^{st} would be sufficient.) Since $\iota(g)$ is an even permutation for all $g \in D_{10}$, irreps V and V' of S_5 restrict to the same representation of D_{10}. We thus omit the rows for V' in our table.

$\iota(\mathcal{C})$	[Id]	[(12345)]	[(12345)]	[(12)(34)]
$\iota^* = \mathrm{Res}^{S_5}_{D_{10}}$	[Id]	$[r]$	$[r^2]$	$[s]$
V^{tr}	1	1	1	1
V^{st}	4	-1	-1	0
$\Lambda^2(V^{st})$	6	1	1	-2
U	5	0	0	1

By inspecting the character table of D_{10}, or using the inner product of characters, one concludes
$$\iota^*(V^{st}) \cong V_2 \oplus V_3.$$

*18. Decompose into irreps the induced representation $\operatorname{Ind}_{D_{10}}^{S_5} V_2$.

Solution. Continuing the computations from the previous question, one gets

$$\iota^*(V^{tr}) \cong \iota^*(V^{\mathbf{sgn}}) \cong V_{tr}, \qquad \iota^*(V^{st}) \cong \iota^*((V^{st})') \cong V_2 \oplus V_3,$$
$$\iota^*(\Lambda^2(V^{st})) \cong V_2 \oplus V_3 \oplus 2V_1, \qquad \iota^*(U) \cong \iota^*(U') \cong V_2 \oplus V_3 \oplus V^{tr}.$$

By Frobenius reciprocity, every $V \in \operatorname{Irrep}(S_5)$ enters into the decomposition of $\operatorname{Ind}_{D_{10}}^{S_5} V_2$ with the same multiplicity with which V_2 enters into the decomposition of $\operatorname{Res}_{D_{10}}^{S_5}(V)$. So

$$\operatorname{Ind}_{D_{10}}^{S_5} V_2 \cong V^{st} \oplus (V^{st})' \oplus \Lambda^2(V^{st}) \oplus U \oplus U'.$$

Double-checking:

$$\dim_{\mathbb{C}}(\operatorname{Ind}_{D_{10}}^{S_5} V_2) = \dim_{\mathbb{C}}(V_2)\frac{\#S_5}{\#D_{10}} = 2\frac{5!}{10} = 24,$$
$$\dim_{\mathbb{C}}(V^{st}) + \dim_{\mathbb{C}}((V^{st})') + \dim_{\mathbb{C}}(\Lambda^2(V^{st})) + \dim_{\mathbb{C}}(U) + \dim_{\mathbb{C}}(U') = 24.$$

S_5	Id	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
V^{tr}	1	1	1	1	1	1	1
$V^{\mathbf{sgn}}$	1	-1	1	-1	1	1	-1
V^{st}	4	2	1	0	-1	0	-1
$(V^{st})'$	4	-2	1	0	-1	0	1
$\Lambda^2(V^{st})$	6	0	0	0	1	-2	0
U	5	1	-1	-1	0	1	1
U'	5	-1	-1	1	0	1	-1