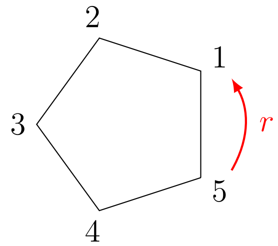
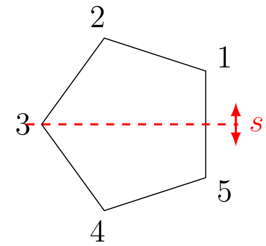
Quiz 2: Character tables

Instructions. Give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those. The question marked with $*$ is a bonus question.

Exercise 1. Let D_{10} be the group of symmetries of a regular pentagon P. Denote by r and by s respectively a $\frac{2\pi}{5}$-rotation and a reflection, as shown in the figure:





You can assume without proof that the following relations hold in D_{10}:

$$r^5 = s^2 = \mathrm{Id}, \qquad srs = r^4,$$

and that the following 10 symmetries form a complete list of the elements of D_{10}:

$$\mathrm{Id}, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4.$$

1. Find an abelian subgroup of D_{10} of index 2.
2. Is D_{10} abelian?
3. What do the previous points tell you about the irreps of D_{10}?
4. Describe all degree 1 representations of D_{10}.
5. Determine the number and the degrees of all irreps of D_{10}.
6. Verify that D_{10} has exactly 4 conjugacy classes:

$$[\mathrm{Id}],\ [r],\ [r^2],\ [s].$$

In what follows, V_1 denotes the non-trivial degree 1 irrep, and V_2 and V_3 the two degree 2 irreps. You can write $\chi_i := \chi^{V_i}$ for simplicity.
First, suppose that

$$V_2 \otimes V_1 \cong V_3.$$

7. What does it mean for the character table?
8. Using the regular representation of D_{10}, complete the character table as far as you can.
9. Looking at the columns of your table, get a contradiction.

We are left with the second possibility:

$$V_2 \otimes V_1 \cong V_2, \qquad V_3 \otimes V_1 \cong V_3.$$

Put $x = \chi_2(r), y = \chi_2(r^2)$.

10. Using the regular representation of D_{10}, express all entries of the character table in terms of x and y.
11. From the orthogonality relation for certain rows, deduce

$$x + y = -1.$$

(Try to pick the easiest pair of rows!)

12. Express the character of the alternating square $\Lambda^2(V_2)$ in terms of x and y.
13. Show that $\Lambda^2(V_2) \cong V_1$. Deduce from this
$$x^2 - y = 2.$$
14. Using Q11 and Q13, determine the values of x and y.
15. Complete the character table of D_{10}.
16. The symmetries of our pentagon P act on its vertices. This yields an injective group morphism $\iota\colon D_{10} \to S_5$ into the symmetric group S_5. Evaluate ι explicitly, without justification, on Id, r, r^2, and s.
17. Using the character table of S_5 recalled below, decompose into irreps the restricted representations $\iota^*(V^{st})$ of D_{10}, where $V^{st} \in \mathrm{Irrep}(S_5)$.

*18. Decompose into irreps the induced representation $\mathrm{Ind}_{D_{10}}^{S_5} V_2$.

S_5	Id	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
V^{tr}	1	1	1	1	1	1	1
$V^{\mathbf{sgn}}$	1	-1	1	-1	1	1	-1
V^{st}	4	2	1	0	-1	0	-1
$(V^{st})'$	4	-2	1	0	-1	0	1
$\Lambda^2(V^{st})$	6	0	0	0	1	-2	0
U	5	1	-1	-1	0	1	1
U'	5	-1	-1	1	0	1	-1