Quiz 1: Representations and characters of finite groups

Exercise 1. Consider the map

$$\rho\colon \mathbb{Z} \to \mathrm{Mat}_{2\times 2}(\mathbb{C}),$$
$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

1. Show that it defines a representation of the group $\mathbb{Z}$ (with the addition operation $+$).
2. What is the degree of ρ?
3. Prove that ρ has precisely one sub-representation of degree 1.
4. Is this sub-representation isomorphic to a representation we have already seen?
5. Give the definition of an irreducible representation. Is ρ irreducible?
6. Give the definition of an indecomposable representation. Is ρ indecomposable?
7. Under what condition is an indecomposable representation of a group G necessarily irreducible?
8. Compute the character of ρ.

Solution.

1. First, the matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ are all invertible. Second, ρ is a group morphism: $\rho(a+b) = \left(\begin{smallmatrix} 1 & a+b \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \rho(a)\rho(b)$. So ρ is a group morphism $\mathbb{Z} \to \mathrm{Mat}^*_{2\times 2}(\mathbb{C})$, that is, a representation.
2. 2, since $\mathrm{Mat}^*_{2\times 2}(\mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^2)$.
3. The 1-dimensional sub-space $\mathbb{C}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is a sub-representation: $a \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Suppose that $(\mathbb{C}^2, \rho)$ has another sub-representation $V = \mathbb{C}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, with $y \neq 0$ (otherwise it coincides with the first one). Then $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) - \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in V$, i.e., $\left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right) \in V$. But $\left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right)$ is not a scalar multiple of $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.
4. It is isomorphic to the trivial representation.
5. An irreducible representation V is a representation having precisely two sub-reps: $\{0\}$ and V itself. Our ρ is not irreducible, since it has a third sub-rep. $\mathbb{C}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.
6. An indecomposable representation V is a representation having precisely two direct summands: $\{0\}$ and V itself. Our ρ is indecomposable: its non-trivial decompositions should have the form $V = V_1 \oplus V_2$, with $\dim_{\mathbb{C}}(V_i) = 1$, but we have proved that it has only one 1-dimensional sub-rep.
7. When G is finite.
8. For all $a \in \mathbb{Z}$, $\chi^\rho(a) = \mathrm{tr}(\rho(a)) = \mathrm{tr}\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) = 2$.

Exercise 2. Recall that for any group G, the $\mathbb{C}$-vector space $\mathbb{C}G$ can be seen as a representation of G in two ways:

- $g \cdot e_h = e_{gh}$ (the left regular representation ρ_{reg});
- $g \cdot e_h = e_{hg^{-1}}$ (the right regular representation ρ_{rreg}).

1. Compute the characters of these representations.
2. Use the result to compare the two representations.

Solution.

1. Both representations are particular cases of permutation reps, so their characters can be computed by counting the fixed points for the corresponding action:
$$\chi^{\rho_{reg}}(g) = \#\{h \in G \,|\, gh = h\} = \#G\delta_{g,1},$$
$$\chi^{\rho_{rreg}}(g) = \#\{h \in G \,|\, hg^{-1} = h\} = \#G\delta_{g,1}.$$
2. The two representations are isomorphic since their characters coincide.

Exercise 3.

1. State Schur's lemma.
2. Consider a representation V of an abelian group G. Show that for any $g \in G$, the map $v \mapsto g \cdot v$ is a G-linear automorphism of V.
3. Assume that G is finite abelian and V is irreducible. Using Schur's lemma and the previous point, show that any $v \in V$ generates a sub-representation $\mathbb{C}v$ of V.
4. Deduce that $V = \mathbb{C}v$ for any $v \in V$, $v \neq 0$.
5. Conclusion: Determine the possible degrees of an irreducible representation of a finite abelian group.

Solution.

1. Let ϕ be a morphism between irreps V, W of a finite group G. Then:
   (a) ϕ is an isomorphism or $\phi = 0$;
   (b) if $V = W$, then $\phi = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{C}$.
2. By the definition of a rep., $v \mapsto g \cdot v$ is a $\mathbb{C}$-linear automorphism of V. Let us show G-linearity: $\forall h \in G$, $g \cdot (h \cdot v) = gh \cdot v = hg \cdot v = h \cdot (g \cdot v)$. We used the definition of reps in terms of G-actions, and the commutativity of G.
3. For any $g \in G$, the G-linear automorphism $v \mapsto g \cdot v$ of our irrep. V should be of the form $v \mapsto \alpha_g v$ for some $\alpha_g \in \mathbb{C}^*$ (Schur's lemma). This means that $\forall v \in V, \forall g \in G$, $g \cdot v = \alpha_g v \in \mathbb{C}v$. So $\mathbb{C}v$ is a sub-rep. of V.
4. Since V is irreducible, its non-zero sub-rep. is necessarily the whole V.
5. By the previous point, it has to be 1.