Quiz 1: Representations and characters of finite groups

Instructions. Don't let the number of questions frighten you – exercises were cut into tiny parts to make them elementary. Give concise but precise answers. When answering a question, you may use the previous questions of the same exercise, even if you have not solved those.

Exercise 1. Consider the map

$$\begin{aligned} \rho\colon \mathbb{Z} &\to \mathrm{Mat}_{2\times 2}(\mathbb{C}), \\ a &\mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

1. Show that it defines a representation of the group $\mathbb{Z}$ (with the addition operation $+$).
2. What is the degree of ρ?
3. Prove that ρ has precisely one sub-representation of degree 1.
4. Is this sub-representation isomorphic to a representation we have already seen?
5. Give the definition of an irreducible representation. Is ρ irreducible?
6. Give the definition of an indecomposable representation. Is ρ indecomposable?
7. Under what condition is an indecomposable representation of a group G necessarily irreducible?
8. Compute the character of ρ.

Exercise 2. Recall that for any group G, the $\mathbb{C}$-vector space $\mathbb{C}G$ can be seen as a representation of G in two ways:

- $g \cdot e_h = e_{gh}$ (the left regular representation ρ_{reg});
- $g \cdot e_h = e_{hg^{-1}}$ (the right regular representation ρ_{rreg}).

1. Compute the characters of these representations.
2. Use the result to compare the two representations.

Exercise 3.

1. State Schur's lemma.
2. Consider a representation V of an abelian group G. Show that for any $g \in G$, the map $v \mapsto g \cdot v$ is a G-linear automorphism of V.
3. Assume that G is finite abelian and V is irreducible. Using Schur's lemma and the previous point, show that any $v \in V$ generates a sub-representation $\mathbb{C}v$ of V.
4. Deduce that $V = \mathbb{C}v$ for any $v \in V$, $v \neq 0$.
5. Conclusion: Determine the possible degrees of an irreducible representation of a finite abelian group.