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ORIGINAL PAPER

Poncelet's theorem and billiard knots

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Abstract Let D be any elliptic right cylinder. We prove that every type of knot can be realized as the trajectory of a ball in D. This proves a conjecture of Lamm and gives a new proof of a conjecture of Jones and Przytycki. We use Jacobi's proof of Poncelet's theorem by means of elliptic functions.

Keywords Poncelet's theorem · Jacobian elliptic functions · Billiard knots · Lissajous knots · Cylinder knots

Mathematics Subject Classification (2000) 57M25

1 Introduction

Jones and Przytycki defined billiard knots as periodic billiard trajectories without self-intersections in a three-dimensional billiard. They proved that billiard knots in a cube are very special knots, the Lissajous knots. They also conjectured that every knot is a billiard knot in some convex polyhedron ([12], see also [4–6,16,22,25]).

Lamm [15,18] proved that not all knots are billiard knots in a cylinder. Then he conjectured that every non-circular elliptic cylinder contains all knots as billiard knots. It is easy to see that Lamm's conjecture implies the conjecture of Jones and Przytycki: if K is a billiard knot in a convex set, then it is also a billiard knot in the polyhedron delimited by the tangent planes. Dehornoy constructed in [8] a billiard which contains all knots, but this billiard is not convex.

In this paper, we obtain a proof of Lamm's conjecture and extend it to links.

Theorem 17 *Let E be an ellipse which is not a circle, and let D be the elliptic cylinder $D = E \times [0, 1]$. Every knot (or link) is a billiard knot (or link) in D.*

Let us give an outline of our proof strategy.

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In Sect. 2 we recall some classical facts about elliptic billiards. If the initial segment of a billiard path in the ellipse E avoids the focal segment of E, then there exists an ellipse C called a caustic, such that the path is circumscribed about C.

We prove that if $n \geq 2k+1$, then there exist n-periodic trajectories winding k times around the caustic. Then, using a theorem on toric braids due to Lamm and Manturov, we deduce that every knot has a planar projection which is a billiard path in an ellipse.

Section 3 presents an explicit proof of the Poncelet porism. This beautiful theorem says that if there is a closed polygon inscribed in a conic E and circumscribed about another conic, then there exist infinitely such polygons, one with a vertex at any given point of E. We recall the Hermite–Laurent proof of Poncelet's theorem by means of Jacobian elliptic functions. The advantage of this proof is that it provides explicit computations of these Poncelet polygons.

Section 4 presents the key step of our strategy. We compute the coordinates of the crossings in terms of Jacobian elliptic functions. Then, some basic holomorphic and arithmetic properties of these functions are used to prove that if n is odd, then there exists a Poncelet polygon of n sides and winding number k, which is totally irregular.

This has the following meaning. Let us suppose that we start at some initial vertex of the polygon and trace out the polygon using arc length. We produce a sequence of distances $t_1, t_2 \ldots$ between the initial point and each successive crossing point. Then the numbers $1, t_1, t_2, \ldots$ are linearly independent over $\mathbf{Q}$.

Section 5 concludes the proof by using the famous density theorem of Kronecker.

If $1, t_1, \ldots, t_k$ *are linearly independent over* $\mathbf{Q}$, *then the set of points* $\big((mt_1), \ldots, (mt_k)\big)$ *is dense in the unit cube, when* m *varies over* $\mathbf{N}$. *Here* (x) *denotes the fractional part of* x.

Now, let $D = E \times [0, 1]$. Let K be the desired knot. Let us start with a copy of K whose planar projection is a totally irregular polygon $\mathcal{P}$. There is a family of periodic billiard paths $\mathcal{P}_m$ which all project to $\mathcal{P}$. One keeps the horizontal component the same and varies the slope m. Using Kronecker's theorem, and adjusting the slope, one can find m such that at the preimages of the crossings, the heights of $\mathcal{P}_m$ are arbitrarily close to some specified list. In particular, one can obtain the over-and-under crossings of $\mathcal{P}_m$ to match those of K. It is remarkable that $\mathcal{P}_m$ may bounce up and down a huge number of times, but this number is invisible to the proof.

There are other applications of Kronecker's theorem to the construction of knots in [13,14].

2 Billiard trajectories in an ellipse

The study of billiard trajectories in an ellipse was introduced by Birkhoff in 1927 [2].

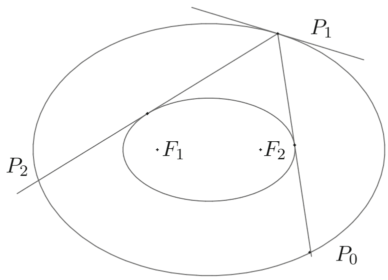
2.1 Some elementary facts

The following results are classical, see [1,9,20,23,27,28] (Fig. 1).

Theorem 1 *Suppose that some segment of a billiard trajectory in an ellipse does not intersect the focal segment* $[F_1 F_2]$. *Then the billiard trajectory remains forever tangent to a fixed confocal ellipse called the caustic.*

Remark 2 When some segment contains one focus, then every segment contains a focus, and there is no caustic. When some segment intersects the interior of the focal segment then there is a caustic, which is a hyperbola with foci F_1 and F_2.

Fig. 1 Existence of an elliptic caustic



Corollary 3 *Let $P_0, P_1, \ldots, P_{n-1}, P_n = P_0$ be a billiard trajectory in an ellipse E such that P_0P_1 does not intersect the focal segment $[F_1F_2]$. Then it is a periodic billiard trajectory inscribed in E and circumscribed about a confocal ellipse C.*

Now, we shall prove the existence of billiard polygons with n sides and rotation number p. Following an idea due to Chasles , Birkhoff [1,2], obtains them as polygons of maximum perimeter among the n-polygons of winding number p inscribed in E. The following simpler result is sufficient for our purposes.

Proposition 4 *Let P_0 be a point of an ellipse E. Let n and p be coprime integers such that $n \geq 2p+1$. There exists a billiard trajectory in E, of period n, of winding number p and starting at P_0.*

Proof Let our ellipse E be defined by its foci F_1 and F_2 and by its major axis $2a$. Let us consider the 1-parameter family of caustics defined by $C_\delta = \{M,\ MF_1 + MF_2 = 2\delta\}$. As it is easy to see, they shrink down to the focal segment $[F_1F_2]$ when $2\delta = F_1F_2$, and expand to the ellipse E if $\delta = a$. When C_δ is the focal segment, the Poncelet trajectory $P_0, P_1, \ldots, P_n$ passes alternately through one focus and then the other. Consequently the winding number ω of this trajectory is greater than $n/2$.

When δ varies from $F_1F_2/2$ to a, the winding number varies continuously from ω to 0. Hence the desired (integral) winding number is achieved. By Corollary 3, this Poncelet polygon is a periodic trajectory. Finally, since p and n are coprime its exact period is n, which concludes the proof. □

Remark 5 This does not prove that the caustics $C_{n,p}$ do not depend on the initial point P_0. This is true by Poncelet's theorem, which we shall prove later.

2.2 Poncelet polygons and toric braids

A toric braid is a braid corresponding to the closed braid obtained by projecting the standardly embedded torus knot into the xy-plane. A toric braid is a braid of the form $\tau_{p,n} = \left(\sigma_1\,\sigma_2\,\ldots\sigma_{p-1}\right)^n$, where $\sigma_1, \ldots, \sigma_{p-1}$ are the standard generators of the full braid group B_p (Fig. 2).

Remark 6 Let E and C be nested ellipses such that there exists a Poncelet polygon inscribed in E and circumscribed about C. Every Poncelet polygon is the projection of a torus knot of type $T(n, p)$, $n \geq 2p+1$. More precisely, if we cut the elliptic annulus delimited by E and C along a half-tangent, then we see that such a polygon is ambient isotopic to the projection of the closure of the toric braid $\tau_{p,n}$. Consequently, it is also ambient isotopic to the star polygon $\left\{{n \atop p}\right\}$, see [14].

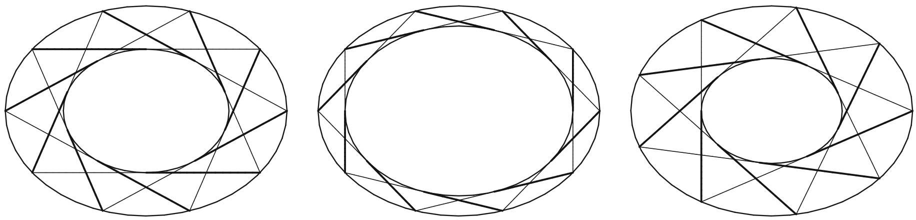


Fig. 2 Some Poncelet polygons (or unions of Poncelet polygons) in nested ellipses. They are projections of the toric braids $\tau_{3,10},\ \tau_{2,10}$ and $\tau_{3,9}$, and are denoted $\left\{{10 \atop 3}\right\}$, $\left\{{10 \atop 2}\right\}$ and $\left\{{9 \atop 3}\right\}$

We shall need the following results on braids, due to Lamm, and independently to Manturov [15,17,21]. A quasitoric braid (or rosette braid) of type $B(p, n)$ is a braid obtained by changing some crossings in the toric braid $\tau_{p,n}$.

The Lamm–Manturov theorem tells us that every knot (or link) is realized as the closure of a quasitoric braid. More precisely, every μ-component link can be realized as the closure of a quasitoric braid of type $B(p\mu, n\mu)$ where $(p, n) = 1$, p even and n odd.

The quasitoric braids form a subgroup of the full braid group, hence there exist trivial quasitoric braids of arbitrarily great length. Consequently, we can suppose $n \geq 2p + 1$ in the Lamm–Manturov theorem. Using this theorem, Proposition 4, and Poncelet's closure theorem, we obtain the main result of this section.

Theorem 7 *Let E be an ellipse. Every μ-component link has a projection which is the union of μ billiard trajectories in E with the same odd period, and with the same caustic C.*

3 Jacobi's proof of Poncelet's theorem

Shortly after the publication of Poncelet's book, Jacobi gave a proof of Poncelet theorem by means of Jacobian elliptic functions [11,24]. We will present the Hermite–Laurent version of Jacobi's proof, apparently forgotten by the experts [19].

The following properties of elliptic functions will be sufficient for our purposes, see [29] for proofs.

3.1 The Jacobian elliptic functions sn z, cn z and dn z

They depend on the choice of a parameter k, $0 < k < 1$, called the elliptic modulus.

The Jacobi amplitude $\varphi = \mathrm{am}(z)$ is defined by inverting the elliptic integral

$$z = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}.$$

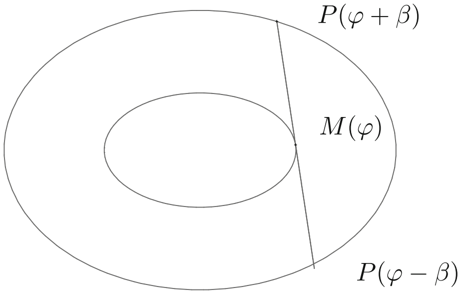
It verifies $\mathrm{am}(u + 2nK) = \mathrm{am}(u) + n\pi$, where $\mathrm{am}(K) = \frac{\pi}{2}$, and $n \in \mathbf{Z}$.

The Jacobian elliptic functions are defined for z real by

$$\mathrm{sn}\, z = \sin\big(\mathrm{am}(z)\big), \quad \mathrm{cn}\, z = \cos\big(\mathrm{am}(z)\big), \quad \mathrm{dn}\, z = \sqrt{1 - k^2 \mathrm{sn}^2 z},$$

and can be extended to meromorphic functions on $\mathbf{C}$. When $k = 0$, these functions degenerate into the ordinary circular function $\sin z$ and $\cos z$. But, contrarily to the circular functions, they are doubly periodic functions with periods $4K \in \mathbf{R}$, and $4iK' \in i\mathbf{R}$, and they have

Fig. 3 Jacobi's Lemma



poles. For example, the poles of $\operatorname{sn} z$ are congruent to iK' (mod.$2K, 2iK'$), its zeros are the points congruent to 0 (mod.$2K, 2iK'$), and its exact periods are $4K,\ 2iK'$. The zeros of $\operatorname{cn} z$ are the points congruent to K (mod.$2K, 2iK'$). We have $\operatorname{sn}(z+2K) = -\operatorname{sn} z$, and $\operatorname{cn}(z+2K) = -\operatorname{cn} z$. We also have $\operatorname{sn}(K+iK') = k^{-1}$, which implies that the zeros of $\operatorname{dn} z$ are the points congruent to $K+iK'$ (mod.$2K, 2iK'$).

We have the following addition formulas

$$\operatorname{sn}(x+y) = \frac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1-k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}, \quad \operatorname{cn}(x+y) = \frac{\operatorname{cn} x \operatorname{cn} y - \operatorname{sn} x \operatorname{sn} y \operatorname{dn} x \operatorname{dn} y}{1-k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}$$

When $k=0$, these formulas degenerate into the usual addition formulas for the circular functions.

In the next section we will use the following formula due to Jacobi [29, p.529].

$$\sin\bigl(\operatorname{am}(u+v)+\operatorname{am}(u-v)\bigr) = \frac{2\operatorname{sn} u \operatorname{cn} u \operatorname{dn} v}{1-k^2 \operatorname{sn}^2 u\ \operatorname{sn}^2 v}$$

3.2 Jacobi's uniformisation

The next result is a variant of Jacobi's uniformization of the Poncelet problem. It is due to Hermite and Laurent [19] (Fig. 3).

Lemma 8 *Let E and C be the ellipses defined by*

$$E = \left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\},\ a>b>1, \quad C = \left\{x^2+y^2=1\right\}.$$

Let us parameterize E by $P(\psi) = (a \operatorname{cn}\psi,\ b\operatorname{sn}\psi)$, and C by $M(\varphi) = (\operatorname{cn}\varphi,\ \operatorname{sn}\varphi)$, where the elliptic modulus k is defined by $k^2(a^2-1) = (a^2-b^2)$. Let β be a real number such that $\operatorname{cn}\beta = 1/a$.

Then the tangent to C at $M(\varphi)$ intersects E at $P(\varphi-\beta)$ and $P(\varphi+\beta)$.

Proof We have $\operatorname{dn}^2\beta = 1-k^2\operatorname{sn}^2\beta = b^2/a^2$, hence $\operatorname{dn}\beta = b/a$.

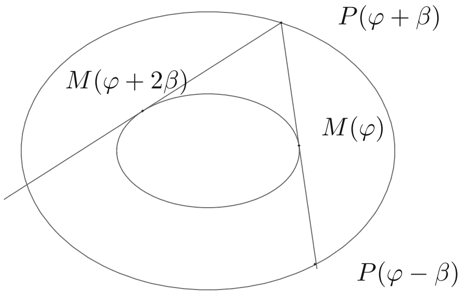
Let us show that $P(\varphi+\beta)$ belongs to the tangent to C at $M(\varphi)$. The equation of this tangent is $x\operatorname{cn}\varphi + y\operatorname{sn}\varphi = 1$. Let us compute $S = a\operatorname{cn}(\varphi+\beta)\operatorname{cn}\varphi + b\operatorname{sn}(\varphi+\beta)\operatorname{sn}\varphi$.

Using the addition formulas we obtain

$$\begin{aligned} S\,(1-k^2\operatorname{sn}^2\varphi\operatorname{sn}^2\beta) &= \operatorname{cn}^2\varphi + \operatorname{sn}^2\varphi\operatorname{dn}^2\beta = \operatorname{cn}^2\varphi + \operatorname{sn}^2\varphi\,(1-k^2\operatorname{sn}^2\beta) \\ &= 1-k^2\operatorname{sn}^2\varphi\operatorname{sn}^2\beta. \end{aligned}$$

Consequently, $S=1$, and $P(\varphi+\beta)$ belongs to the tangent to C at $M(\varphi)$. Changing β to $-\beta$, we see that $P(\varphi-\beta)$ also belongs to this tangent. □

Fig. 4 Proof of Poncelet's closure theorem



Remark 9 By affinity, Jacobi's lemma extends easily in the case of two nested ellipses with the same two axis, meeting transversally in $P_2(\mathbf{C})$. When this pair of ellipses becomes affinely equivalent to a pair of concentric circles, the elliptic parametrizations degenerate into the usual circular ones.

3.3 Proof of Poncelet's closure theorem

We shall now present the Hermite–Laurent proof of Poncelet's theorem for a pair of confocal ellipses. Since any pair of conics meeting transversally in $P_2(\mathbf{C})$ is projectively equivalent to a pair of confocal ellipses [20], we obtain a proof of the generic case of Poncelet's theorem. For the nongeneric cases see [1,9,26], and for the original proof of Jacobi see [3] (Fig. 4).

Proof of Poncelet's closure theorem. Let $P_0, P_1, \ldots P_{n-1} P_0$ be a Poncelet polygon inscribed in E and circumscribed about C. Let $P_j P_{j+1}$ be tangent to C at M_j. We will use the Jacobi parametrizations of E and C. If $M_0 = M(\varphi)$, then by Jacobi's lemma we can suppose $P_1 = P(\varphi+\beta)$. Using Jacobi's lemma again, we have $M_1 = M(\varphi+2\beta)$, and by induction $M_j = M(\varphi+2j\beta)$.

Since the polygon closes after n steps, we have $M_n = M_0$, or $M(\varphi+2n\beta) = M(\varphi)$.

That means $\mathrm{am}(\varphi+2n\beta) = \mathrm{am}\,\varphi + 2q\pi = \mathrm{am}(\varphi+4qK)$ by the properties of the Jacobi amplitude. Consequently we obtain $2n\beta = 4qK$, or $\beta = 2qK/n$.

Now, let us consider a Poncelet polygon starting from an arbitrary point $M'_0 = M(\varphi')$ of C. By Jacobi's lemma we have $M'_n = M(\varphi'+2n\beta) = M(\varphi'+4qK) = M(\varphi') = M'_0$.

Consequently, we see that every Poncelet polygonal line closes after n steps. □

Remark 10 Darboux has shown that if $\mathcal{P}$ is a Poncelet polygon with an even number of sides in two confocal ellipses, then $\mathcal{P}$ possesses a central symmetry. This result shows that a totally irregular Poncelet polygon has necessarily an odd number of sides, see [7,23].

4 Irregularity of Poncelet odd polygons

Most regularity properties of a polygon can be expressed by rational linear relations between some of its segments. Let us parameterize a (crossed) polygon by arc length, starting at a vertex P_0. We shall say that this polygon is totally irregular if 1 and the arc lengths of its crossings and vertices (except P_0) are linearly independent over $\mathbf{Q}$.

The purpose of this section is to prove that if E and C is a pair of confocal ellipses possessing a Poncelet polygon with an odd number of sides, then there exists a totally irregular Poncelet polygon. We will give an analogous result for unions of finitely many Poncelet polygons.

4.1 Two lemmas on elliptic functions

We shall use elliptic functions to compute the arc lengths of the crossings and vertices of Poncelet polygons. We shall need the following two technical lemmas.

Lemma 11 *Let n and p be coprime integers, with n odd. For every integer j, let us define the function $f_j(z) = \mathrm{sn}^2(z + j\theta) + r^2$, where $r^2 > 0$ and $\theta = 4pK/n$. Then, if $h \not\equiv j \pmod{n}$, the functions $f_j(z)$ and $f_h(z)$ do not possess any common zero.*

Proof First, let us study the zeros of the elliptic function $g(z) = \mathrm{sn}\, z + ir$, $r > 0$. By considering its restriction to the y-axis, we see that there exists a pure imaginary α, such that $g(\alpha) = 0$. Since we have $\mathrm{sn}(2K - \alpha) = \mathrm{sn}\,\alpha$, we see that $2K - \alpha$ is another zero of $g(z)$. As $g(z)$ is an elliptic function of order two, its zeros are the points congruent to α or $2K - \alpha$ (mod. $4K, 2iK'$). By parity, we deduce that the zeros of $f_j(z)$ are the numbers which are congruent to $\pm\alpha - j\theta$, or $2K \pm \alpha - j\theta$, (mod. $4K, 2iK'$).

If we had $\alpha - j\theta \equiv \alpha - h\theta$, or $\alpha - j\theta \equiv 2K + \alpha - h\theta$ (mod.$4K$, $2iK'$), then we would deduce $(h - j)\theta \equiv 0 \pmod{2K}$. This implies that $2(h - j)p/n$ is an integer, which is impossible since n is odd, $(n, p) = 1$, and $h \not\equiv j \pmod{n}$.

If we had $\alpha - j\theta \equiv -\alpha - h\theta$ or $\alpha - j\theta \equiv 2K - \alpha - h\theta$, (mod.$4K$, $2iK'$), then we would have $2\alpha \equiv ((j - h)\theta$ (mod. $2K, 2iK'$). Taking the real parts, we would obtain $(j - h)\theta \equiv 0 \pmod{2K}$ which is impossible.

Consequently, $\alpha - j\theta$ cannot be a zero of $f_h(z)$. The proof that the other zeros of $f_j(z)$ cannot be zeros of $f_h(z)$ is entirely similar. □

Remark 12 As the proof shows it, the condition n odd is necessary in Lemma 11.

Lemma 13 *Let n and p be coprime integers. For $j \not\equiv 0 \pmod{n}$, let us define the functions $D_j(z)$ and $F_j(z)$ by*

$$D_j(z) = \mathrm{sn}(z + j\theta)\,\mathrm{cn}\, z - \mathrm{cn}(z + j\theta)\,\mathrm{sn}\, z, \quad F_j(z) = \frac{\mathrm{sn}(z + j\theta) - \mathrm{sn}\, z}{D_j(z)}, \ \text{where}\ \theta = \frac{4pK}{n}.$$

Then, for every integer j there exists a complex number α_j such that $F_j(\alpha_j) = \infty$, and $F_h(\alpha_j) \neq \infty$. for $h \not\equiv j \pmod{n}$.

Proof We have

$$D_j(z) = \sin\big(\mathrm{am}(z + j\theta) - \mathrm{am}\, z\big) = \sin\big(\mathrm{am}(z + j\theta) + \mathrm{am}(-z)\big)$$

Now, using the Jacobi formula for $\sin\big(\mathrm{am}(u + v) + \mathrm{am}(u - v)\big)$, we obtain

$$D_j(z) = \frac{2\,\mathrm{sn}(j\beta)\,\mathrm{cn}(j\beta)\,\mathrm{dn}(z + j\beta)}{1 - k^2\,\mathrm{sn}^2(j\beta)\,\mathrm{sn}^2(z + j\beta)}, \quad \text{where } \beta = \frac{\theta}{2}$$

Let $\alpha_j = -j\beta + K + iK'$. We have $\mathrm{dn}(\alpha_j + j\beta) = \mathrm{dn}(K + iK') = 0$. Since $\mathrm{dn}^2 z + k^2\,\mathrm{sn}^2 z = 1$, we obtain $\mathrm{sn}^2(\alpha_j + j\beta) = 1/k^2$, and then $D_j(\alpha_j) = 0$.

The numerator of $F_j(\alpha_j)$ is

$$N(\alpha_j) = \mathrm{sn}(\alpha_j + j\theta) - \mathrm{sn}(\alpha_j) = \mathrm{sn}(K + iK' + j\beta) - \mathrm{sn}(K + iK' - j\beta).$$

Using the addition formula for the function $\mathrm{sn}\, z$, we obtain

$$N(\alpha_j) = 2\frac{\mathrm{sn}(K + iK')\,\mathrm{cn}(j\beta)\,\mathrm{dn}(j\beta)}{1 - k^2\,\mathrm{sn}^2(K + iK')\,\mathrm{sn}^2(j\beta)}.$$

Since $\mathrm{sn}(K + iK') = k^{-1}$, we obtain $N(\alpha_j) = 2k^{-1}\dfrac{\mathrm{dn}(j\beta)}{\mathrm{cn}(j\beta)} \neq 0$, and then $F_j(\alpha_j) = \infty$.

On the other hand, if $h \not\equiv j \pmod n$, we have $\alpha_j + h\beta = K + iK' + 2(h-j)pK/n$.

First, we see that $\alpha_j + h\beta \not\equiv K + iK' \ (\mathrm{mod}.2K, 2iK')$, which implies that $\mathrm{dn}(\alpha_j + h\beta) \neq 0$.

We also see that $\alpha_j + h\beta \not\equiv iK' \ (\mathrm{mod}.2K, 2iK')$, which implies that $\mathrm{sn}(\alpha_j + h\beta) \neq \infty$. We conclude that $D_h(\alpha_j) \neq 0$.

Let us show that if $\mathrm{sn}\, z = \infty$, then $F_h(z) \neq \infty$.

Since the functions $\mathrm{sn}\, z$ and $\mathrm{sn}(z + h\theta)$ do not have common poles, $\mathrm{sn}(z + h\theta) \neq \infty$.

On the other hand, as $\mathrm{sn}^2 z + \mathrm{cn}^2 z = 1$, we obtain

$$\frac{\mathrm{cn}^2 z}{\mathrm{sn}^2 z} = -1, \text{ and then } F_h(z) = \frac{-1}{\mathrm{sn}(z + h\theta)\dfrac{\mathrm{cn}\, z}{\mathrm{sn}\, z} - \mathrm{cn}(z + h\theta)}$$

If we had $F_h(z) = \infty$, then

$$\mathrm{sn}(\mathrm{z} + \mathrm{h}\theta)\frac{\mathrm{cn}\, \mathrm{z}}{\mathrm{sn}\, \mathrm{z}} = \mathrm{cn}(\mathrm{z} + \mathrm{h}\theta),$$

whence $\mathrm{sn}^2(z + h\theta) = -\mathrm{cn}^2(z + h\theta) \neq \infty$, and $\mathrm{sn}^2(z + h\theta) + \mathrm{cn}^2(z + h\theta) = 0$, which is impossible.

Similarly, we see that if $\mathrm{sn}(z + h\theta) = \infty$, then $F_h(z) \neq \infty$.

Now, let us prove that $F_h(\alpha_j) \neq \infty$. We have $D_h(\alpha_j) \neq 0$, and we have proved that we can suppose $\mathrm{sn}(\alpha_j) \neq \infty$ and $\mathrm{sn}(\alpha_j + h\theta) \neq \infty$, then

$$F_h(\alpha_j) = \frac{\mathrm{sn}(\alpha_j + h\theta) - \mathrm{sn}\, \alpha_j}{D_h(\alpha_j)} \neq \infty.$$

□

4.2 Irregular Poncelet polygons with an odd number of sides

Proposition 14 *Let E and C be confocal ellipses such that there exists a Poncelet polygon $\mathcal{P}$ inscribed in E and circumscribed about C. We suppose that the number of sides of $\mathcal{P}$ is odd. Then there exists a Poncelet polygon satisfying the following condition.*

If the arc lengths t_i of the vertices and crossings are measured from a vertex P_0, then the numbers 1 and t_i, $t_i \neq 0$ are linearly independent over $\mathbf{Q}$.

Proof

$$\text{Let } E = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\},\ a > b > 1, \text{ and } C = \{x^2 + \frac{y^2}{c^2} = 1\},\ c < 1$$

be our ellipses.

Let us consider the Jacobi parametrizations of E and C by means of elliptic functions, and let $\theta = 4pK/n$. To each real number φ corresponds a Poncelet polygon $\mathcal{P}_\varphi$ through $M(\varphi) = (\mathrm{cn}\,\varphi,\ c\,\mathrm{sn}\,\varphi)$. Let us denote $\varphi_j = \varphi + j\theta$, $M_j = M(\varphi_j)$, and let ℓ_j be the tangent to C at M_j. The equation of ℓ_j is

$$x\,\mathrm{cn}\,\varphi_j + \frac{y}{c}\,\mathrm{sn}\,\varphi_j = 1.$$

Let $Q_{h,j} = \ell_h \bigcap \ell_{h+j}$, $j \not\equiv 0 \pmod n$. The abcissa $x_{h,j}$ of $Q_{h,j}$ is

$$x_{h,j} = \frac{-\,\mathrm{sn}\,\varphi_h + \mathrm{sn}(\varphi_h + j\theta)}{\mathrm{sn}(\varphi_h + j\theta)\,\mathrm{cn}\,\varphi_h - \mathrm{cn}(\varphi_h + j\theta)\mathrm{sn}\,\varphi_h} = F_j(\varphi_h)$$

where F_j is the function defined in Lemma 13. The abcissa of $P_h = Q_{h,-1} = Q_{h-1,1}$ will also be denoted by $x_h = x_{h,-1}$. The distance $P_h Q_{h,j}$ is $|d_{h,j}|$ where

$$d_{h,j} = d_{h,j}(\varphi) = \frac{\sqrt{1-c^2}}{\operatorname{sn}\varphi_h}\sqrt{\operatorname{sn}^2\varphi_h + \frac{c^2}{1-c^2}}\left(x_h - x_{h,j}\right)$$

Since $c^2/(1-c^2) > 1$, the function $d_{h,j}(\varphi)$ is meromorphic in a neighborhood of the real axis.

Our first step is to prove that the functions 1 and $d_{h,j}(\varphi)$, $j \not\equiv -1 \pmod n$ are linearly independent over $\mathbf{C}$.

Let $\lambda_{h,j}$ and λ be complex numbers such that $\sum_{h=1}^{n}\sum_{j=1}^{n-2}\lambda_{h,j}d_{h,j} = \lambda$, or

$$\sum_{h=1}^{n}\frac{\sqrt{1-c^2}}{\operatorname{sn}\varphi_h}\sqrt{\operatorname{sn}^2\varphi_h + \frac{c^2}{1-c^2}}\left(\sum_{j=1}^{n-2}\lambda_{h,j}(x_h - x_{h,j})\right) = \lambda.$$

Since $c^2/(1-c^2) > 0$, we see by Lemma 11 that the functions $f_h(\varphi) = \sqrt{\operatorname{sn}^2\varphi_h + c^2/(1-c^2)}$ do not possess any common zero. Hence, in the neighborhood of a zero of $f_h(\varphi)$ this function is not meromorphic, while the other functions are.

This implies that for every $h = 1\ldots n$ we have

$$\sum_{j=1}^{n-2}\lambda_{h,j}\left(x_h - x_{h,j}\right) = 0, \text{ and then } \lambda = 0.$$

Using our expressions of the abcissas $x_{h,j}$, we obtain the following relation between meromorphic functions

$$\sum_{j=1}^{n-2}\lambda_{h,j}\left(F_{-1}(z) - F_j(z)\right) = 0.$$

By Lemma 13, for every integer $j \neq 0$ there exists a number α_j such that $F_j(\alpha_j) = \infty$, and $F_h(\alpha_j) \neq \infty$ if $h \not\equiv j \pmod n$. Letting $z = \alpha_j$, we obtain $\lambda_{h,j} = 0$, which concludes the proof of the linear independence of our functions.

Now, we shall prove that for most $\varphi \in \mathbf{R}$, the numbers $d_{h,j}(\varphi)$ and 1 are linearly independent over $\mathbf{Q}$.

For every nonzero collection of rational numbers $\Lambda = (\lambda, \lambda_{h,j})$, let us define the function F_Λ by $F_\Lambda(\varphi) = \lambda - \sum_{h,j}\lambda_{h,j}\, d_{h,j}(\varphi)$. By our first step, this function is not identically zero, and it is meromorphic in a neighborhood of $\mathbf{R}$. Therefore, the set of its real zeros is countable. Consequently, the set of all real numbers φ such that 1 and the numbers $d_{h,j}(\varphi)$ are linearly dependent over $\mathbf{Q}$ is countable. By cardinality, we deduce that the complementary set is not countable, hence nonempty. Consequently, there exists a real φ such that 1 and the numbers $|d_{h,j}(\varphi)|$ are linearly independent over $\mathbf{Q}$.

Now, let us parameterize our Poncelet polygon by arc length, starting from P_0 for $t_0 \in \mathbf{Q}$. The arc length $t_{h,j}$ of $Q_{h,j}$ is

$$\begin{aligned} t_{h,j} &= t_0 + d(P_0, P_1) + d(P_1, P_2) + \cdots + d(P_{h-1}, P_h) + d(P_h, Q_{h,j}) \\ &= t_0 + |d_{0,1}| + |d_{1,1}| + |d_{2,1}| + \cdots + |d_{h-1,1}| + |d_{h,j}|. \end{aligned}$$

The result follows from the independence of the numbers 1 and $|d_{h,j}|$. □

We shall also need an analogous result for links.

Proposition 15 *Let E and C be confocal ellipses such that there exists a polygon of an odd number of sides inscribed in E and circumscribed about C.*

For any integer μ, there exist μ Poncelet polygons $\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(\mu-1)}$ satisfying the following condition:

for each such polygon, if t_i are the arc lengths corresponding to its vertices , its crossings, an its intersections with the other polygons, then the numbers 1 and $t_i,\ i \neq 0$ are linearly independent over $\mathbf{Q}$.

Proof Let $\tau = \dfrac{\theta}{\mu}$, and let us denote $M_h = M(\varphi + h\tau) \in C$, and ℓ_h the tangent to C at M_h. Let us consider the Poncelet polygons $\mathcal{P} = \mathcal{P}^{(0)},\ \mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(\mu-1)}$ through the points $M_0, M_1, \ldots, M_{\mu-1}$. The polygon $\mathcal{P}$ is tangent to C at the points $M_0, M_\mu, M_{2\mu}, \ldots, M_{(n-1)\mu}$. The vertices and crossings of $\mathcal{P}$ are the points $Q_{h,j} = \ell_h \bigcap \ell_{h+j}$, where $h \equiv 0 \pmod{\mu}$.

Just as before, it can be proved that the distances 1 and $|d_{h,j}(\varphi)|,\ h \equiv 0,\ j \not\equiv 0,\ j \not\equiv -1 \pmod{\mu}$ are linearly independent over $\mathbf{Q}$, except for a countable set of numbers φ.

Consequently, the number 1 and the arc lengths $t_i,\ i \neq 0$ of the crossings and vertices of $\mathcal{P}$ are linearly independent over $\mathbf{Q}$ except on a countable set of values of φ.

By cardinality, we can suppose that the same property is true for each polygon $\mathcal{P}^{(j)}$, $j = 0, \ldots, \mu - 1$, which proves our result. □

5 Proof of the theorem

We will use Kronecker's theorem (see [10, Theorem 443]):

Theorem 16 *If $\theta_1, \theta_2, \ldots, \theta_k, 1$ are linearly independent over $\mathbf{Q}$, then the set of points $\big((m\theta_1), \ldots, (m\theta_k)\big)$ is dense in the unit cube, when m varies over $\mathbf{N}$. Here (x) denotes the fractional part of x.*

Now, we can prove our main theorem.

Theorem 17 *Let E be an ellipse which is not a circle, and let D be the elliptic cylinder $D = E \times [0, 1]$. Every knot (or link) is a billiard knot (or link) in D.*

Proof First, we consider knots. By Theorem 7 there exists a knot isotopic to K, whose projection on the xy-plane is a billiard trajectory of odd period in the ellipse E. If $t_0, t_1, \ldots, t_k$ are the arc lengths corresponding to the vertices and crossings, we can suppose by Proposition 14 that the numbers $t_1, \ldots, t_k$, and 1 are linearly independent over $\mathbf{Q}$. Rescaling if necessary, we can suppose that the total length of the trajectory is 1.

Let us consider the polygonal curve defined by $(x(t), y(t), z(t))$, where $z(t)$ is the sawtooth function $z(t) = 2|(mt + \varphi) - 1/2|$ depending on the integer m and on the real number φ. If the heights $z(P_j)$ of the vertices are such that $z(P_j) \neq 0,\ z(P_j) \neq 1$, then it is a periodic billiard trajectory in the elliptic cylinder $\mathbf{D} = E \times [0, 1]$ (see [12,13,16,18,25]). If we set $\varphi = 1/2 + z_0/2,\ z_0 \in (0, 1)$, we have $z(0) = z_0$. Now, using Kronecker's theorem, there exists an integer m such that the numbers $z(t_i)$ are arbitrarily close to any specified collection of heights, which completes our proof.

The case of μ-component links is similar. First, by Theorem 7, we find a diagram that is the union of μ Poncelet polygons with the same odd number of sides. Then, by Proposition 15 and Kronecker's theorem, we parameterize each component so that the heights of the vertices and crossings are close to any specified list. □

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