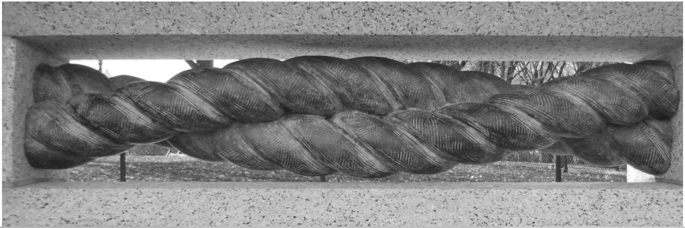
Qualgebras and knotted 3-valent graphs

Victoria LEBED
work in progress with ***Seiichi KAMADA***

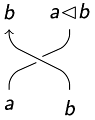
OCAMI, Osaka

Knots in Washington XXXVII
January 19-20, 2014



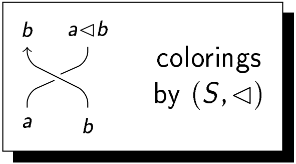
Part 1:
How a Knot Theorist Would Invent Qualgebras

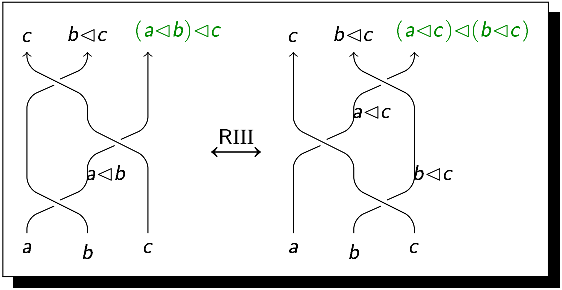
Quandle colorings of knots



colorings by $(S, \lhd)$

Quandle colorings of knots





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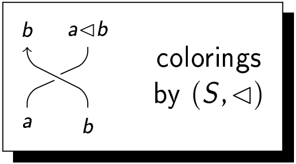

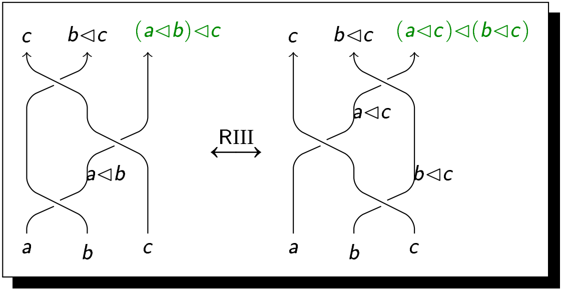

RIII $\leftrightarrow$ $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ (SD)

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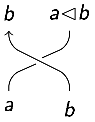
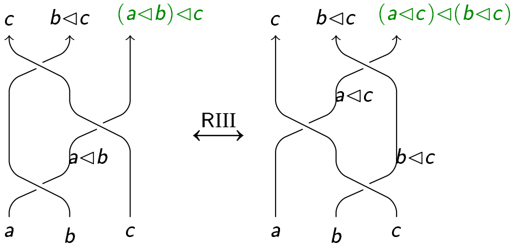


colorings by $(S, \triangleleft)$



RIII	$\leftrightarrow$	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	(SD)
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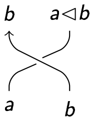

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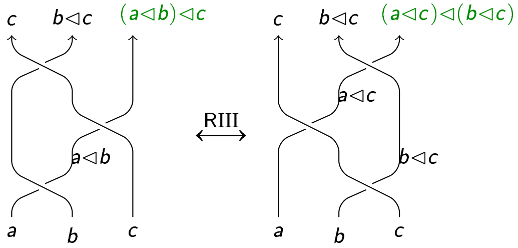

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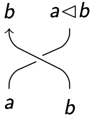
Example

Group $G \rightsquigarrow (G, g \triangleleft h = h^{-1}gh)$.

Quandle colorings of knots



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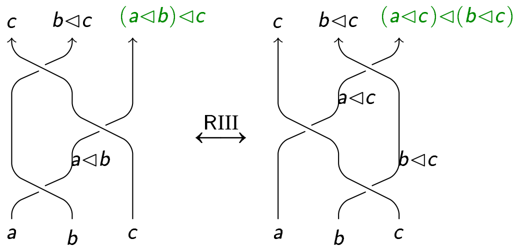

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knot invariants $\overset{\text{colorings}}{\rightsquigarrow}$ quandle

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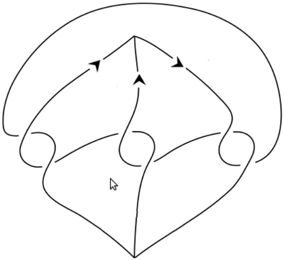
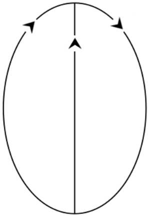
Knotted 3-valent graphs

Kinoshita-Terasaka Θ-curve

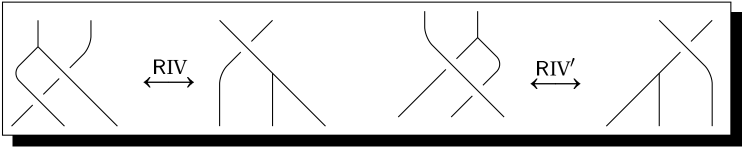


standard Θ-curve



Reidemeister moves for knotted 3-valent graphs

1989 L.H.Kauffman, S.Yamada, D.N.Yetter:



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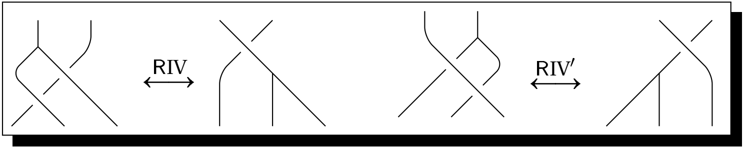
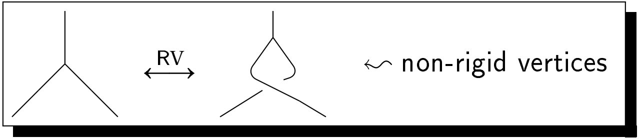
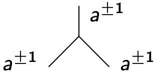




Extending quandle colorings to 3-valent graphs?

Approaches:

❀ 

G is a group; $a^3 = 1$

(1995 C.Livingston; a vertex constant version of $\pi_1(\mathbb{R}^3 \backslash \Gamma)$)

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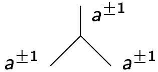
❀ 

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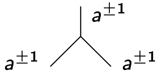
❀ 

S is a symmetric quandle; $a^{+1} = a, a^{-1} = \overline{a}$; $\forall x \in S,\ ((x \lhd a) \lhd a) \lhd a = x$

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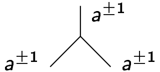
- 

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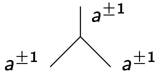
- 

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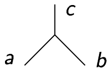
- 

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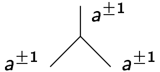
- 

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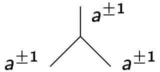
- 

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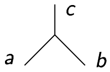
- 

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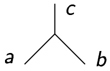
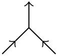
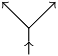
- 

S is a quandle; $\forall x \in S,\ ((x \lhd^{\pm 1} a) \lhd^{\pm 1} b) \lhd^{\pm 1} c = x$
(2010 M.Niebrzydowski)

Orientation

Well-oriented 3-graphs: only zip  and unzip  vertices.

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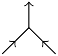
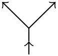
Well-oriented 3-graphs: only zip  and unzip  vertices.

WinZip 2013

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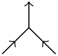
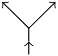
Well-oriented 3-graphs: only zip  and unzip  vertices.

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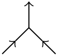
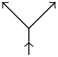
Well-oriented 3-graphs: only zip  and unzip  vertices.

Proposition: Every 3-graph admits a well-orientation.

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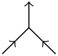
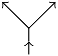
Well-oriented 3-graphs: only zip  and unzip  vertices.

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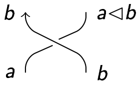
Unoriented 3-graph $\longmapsto$ { its well-orientations }.

Qualgebra colorings for knotted 3-valent graphs



colorings by $(S, \lhd, \blacklozenge)$

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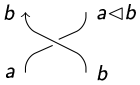

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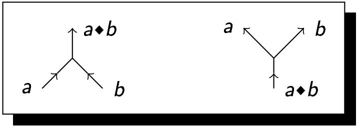

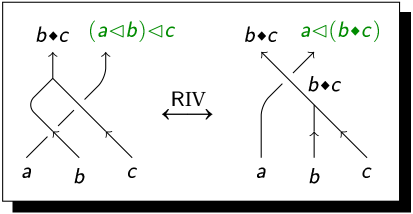

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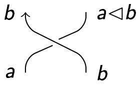

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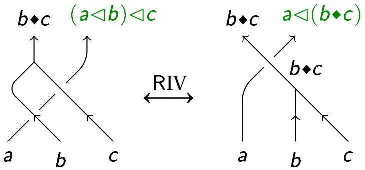

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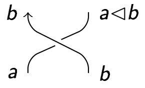

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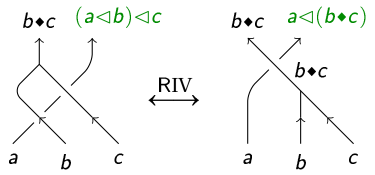

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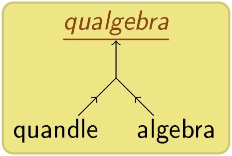

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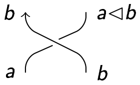

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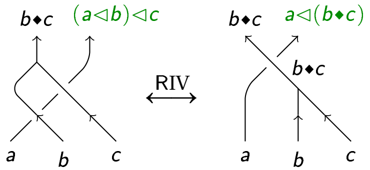

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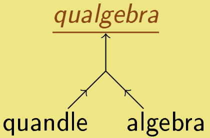
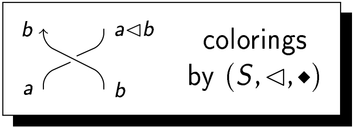
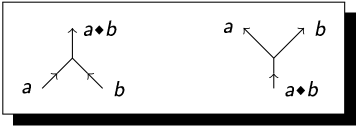
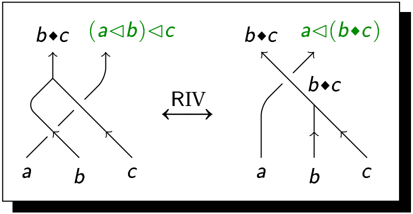


3-graph invariants $\overset{\text{colorings}}{\rightsquigarrow}$ qualgebra

Qualgebra colorings for knotted 3-valent graphs







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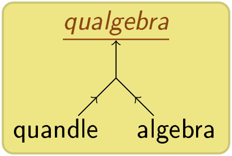

Related construction

1986 P.Dehornoy: augmented RDS & RD-monoids;
braided Thompson groups (braids with “close” strands).

Part 2: How an Algebraist Would Invent Qualgebras

Group qualgebras

Example 1

Group $G \rightsquigarrow$ *group qualgebra* $QA(G) := (G, g \lhd h = h^{-1}gh, g \blacklozenge h = gh)$.

Group qualgebras

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$$QA(G)\text{-colorings} \xleftrightarrow[\text{presentation}]{\text{Wirtinger}} \mathrm{Hom}(\pi_1(\mathbb{R}^3 \backslash K), G)$$

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abstract level	quandle axioms
group level	conjugation

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abstract level	quandle axioms	qualgebra axioms
group level	conjugation	conjugation/multiplication interaction

Group qualgebras

Example 1

Group $G \rightsquigarrow$ *group qualgebra* $QA(G) := (G, g \lhd h = h^{-1}gh, g \blacklozenge h = gh)$.

$$QA(G)\text{-colorings} \xleftrightarrow[\text{presentation}]{\text{Wirtinger}} \mathrm{Hom}(\pi_1(\mathbb{R}^3 \backslash K), G)$$

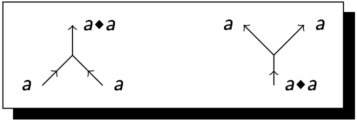
abstract level	quandle axioms	qualgebra axioms
group level	conjugation	conjugation/multiplication interaction

Example 1'

Group G & subset $X \subset G$ $\rightsquigarrow$ the sub-qualgebra of $QA(G)$ generated by X.

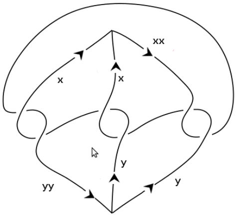
Computation example

Isosceles colorings:

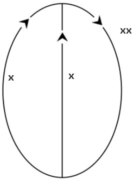


Computation example

Kinoshita-Terasaka Θ-curve



standard Θ-curve



$$\#\{\text{isosceles } Q-\text{colorings of } \Theta\} = \#Q$$

$$\#\{\text{isosceles } Q-\text{colorings of } \Theta_{KT}\} = \#\{(x,y) \in Q \times Q | (*)\}$$

$$\left.\begin{array}{rcl} (y \diamond y)\widetilde{\lhd}x & = & (x \diamond x) \lhd y \\ x\widetilde{\lhd}(y \diamond y) & = & y\widetilde{\lhd}x \\ x \lhd y & = & y \lhd (x \diamond x) \end{array}\right\} (*)$$

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Pairs (x,x) satisfy $(*)$.
We look for $Q(G)$-colorings with $x \neq y$.

Computation example

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Try $x^3=y^3=1$: $(*)\Leftrightarrow(xyx=yxy)$.

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In S_4, take $x = (123)$, $y = (432)$: one has $xyx = yxy = (14)(23)$.

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Conclusion:

$$\#\{\text{isosceles } Q(S_4)-\text{colorings of } \Theta\} = \#Q,$$

$$\#\{\text{isosceles } Q(S_4)-\text{colorings of } \Theta_{KT}\} > \#Q.$$

Trivial and free examples

Example 0

$a \lhd b = a$ and any commutative $\diamond$

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$a \lhd b = a$ and any commutative $\diamond$ $\rightsquigarrow$ abstract graph invariants.

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Example ∞

Quandle $Q \rightsquigarrow$ an associative qualgebra $(S(Q), \lhd, \text{concat})$.

$$S(Q) = \bigsqcup_{n \geqslant 1} Q^n \Big/ (\ldots, a, b, \ldots) = (\ldots, b, a \lhd b, \ldots)$$

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Question: Which quandles can be "qualgebralized"?

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Question: Which quandles can be "qualgebralized"?
Negative example: dihedral quandle $(\mathbb{Z}_n, a \lhd b = 2b - a)$.

$$(a \lhd b) \lhd c = 2c - 2b + a$$
$$a \lhd (b \blacklozenge c) = 2(b \blacklozenge c) - a$$

Small examples

Example 4

Non-trivial qualgebra structures on $Q = \{p, q, r, s\}$?

Description: put $\overline{p} = q, \overline{q} = p, \overline{r} = r, \overline{s} = s$;

$x \lhd r = \overline{x}, \qquad x \lhd y = x$ for other y;

$\blacklozenge$ is commutative,

$r \blacklozenge x = r$ for $x \neq r, \qquad r \blacklozenge r = s \blacklozenge s = s$,

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❀ Two are associative: $p \blacklozenge s = q \blacklozenge s = s$, $p \blacklozenge p = \overline{q \blacklozenge q} \in \{p, s\}$.

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❀ Three have neutral elements.

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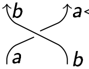
❀ Three have neutral elements.

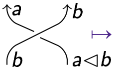
❀ None are unital associative.

Part 3:
Towards Qualgebra Homology

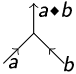
Qualgebra cocycle invariants for knotted 3-valent graphs



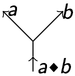
$\mapsto \chi(a,b)$



$\mapsto -\chi(a,b)$



$\mapsto \lambda(a,b)$



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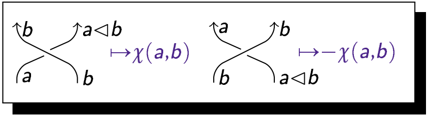
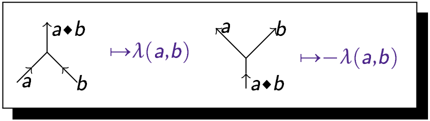
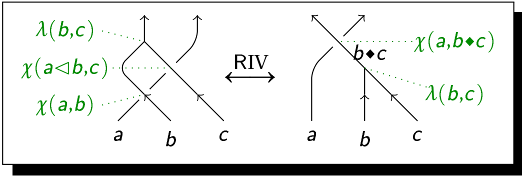
Qualgebra cocycle invariants for knotted 3-valent graphs







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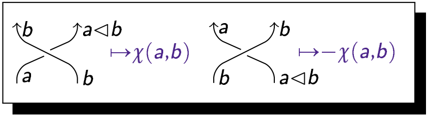
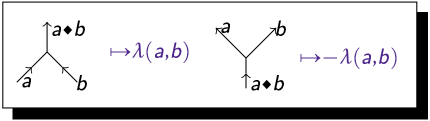
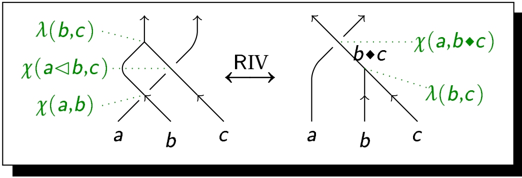
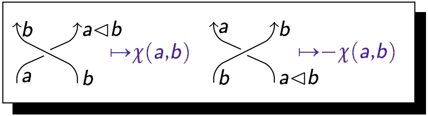


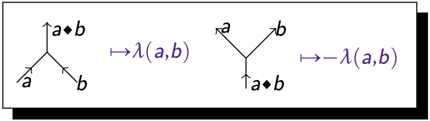


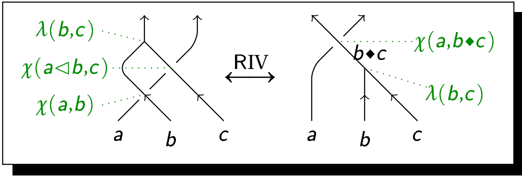


RIV	$\leftrightarrow$	$\chi(a,b \blacklozenge c) = \chi(a,b) + \chi(a \triangleleft b,c)$
RIV′	$\leftrightarrow$	$\chi(a \blacklozenge b,c) + \lambda(a,b) =$ $\chi(a,c) + \chi(b,c) + \lambda(a \triangleleft c, b \triangleleft c)$
RV	$\leftrightarrow$	$\chi(a,b) + \lambda(b, a \triangleleft b) = \lambda(a,b)$
RIII	$\leftrightarrow$	$\chi(a,b) + \chi(a \triangleleft b,c) = \chi(a \triangleleft c, b \triangleleft c) + \chi(a,c)$
RI	$\leftrightarrow$	$\chi(a,a) = 0$

Qualgebra cocycle invariants for knotted 3-valent graphs







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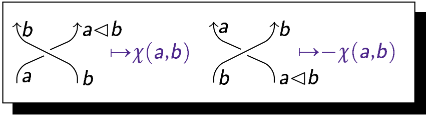
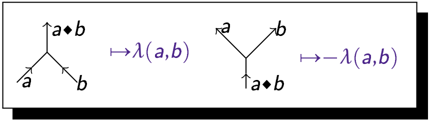
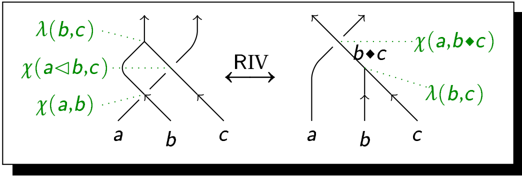
} *qualgebra 2-cocycle*

Qualgebra cocycle invariants for knotted 3-valent graphs







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} *qualgebra* *2-cocycle*

3-graph invariants $\overset{\text{colorings}}{\underset{\text{weights}}{\leftsquigarrow}}$ qualgebra & 2-cocycle

Qualgebra cocycles: example

$Q = \{p, q, r, s\}$

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$\blacklozenge$ is commutative,

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The space of 2-cocycles is 8-dimensional:

$$\chi(p,r) + \chi(q,r) = 0, \qquad \chi(x,y) = 0 \text{ for other } (x,y),$$

$$\lambda(x,y) = \lambda(y,x), \qquad \lambda(p,r) - \lambda(q,r) = \chi(p,r),$$

$$\lambda(p,p) - \lambda(q,q) = 2\chi(p,r) - \chi(p \blacklozenge p, r),$$

$$\lambda(p,s) - \lambda(q,s) = \chi(p,r) - \chi(p \blacklozenge s, r).$$

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The space of 2-coboundaries is 4-dimensional.

Qualgebra cocycle invariants: diverse questions

Two-coboundaries

$$\phi : S \to \mathbb{Z} \rightsquigarrow \qquad \begin{aligned} \chi(a,b) &= \phi(a \lhd b) - \phi(a), \\ \lambda(a,b) &= \phi(a \blacklozenge b) - \phi(a) - \phi(b). \end{aligned}$$

Qualgebra cocycle invariants: diverse questions

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Question: Define and study qualgebra homology in higher degrees?

Qualgebra cocycle invariants: diverse questions

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Question: Define and study qualgebra homology in higher degrees?

Enhancements

❀ Region coloring and shadow cocycle invariants.

Qualgebra cocycle invariants: diverse questions

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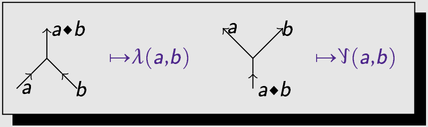
❀Region coloring and shadow cocycle invariants.

❀Distinguish zip- and unzip-vertices:



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