Laver Tables: from Set Theory to Braid Theory

(Based on joint work with Patrick Dehornoy)

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Abstract

Laver tables A_n are certain finite shelves (i.e., sets endowed with a binary operation distributive with respect to itself). They originate from Set Theory and, in spite of an elementary definition, have complicated combinatorial properties. They are conjectured to approximate the free monogenerated shelf $\mathcal{F}_1$, this conjecture being currently proved only under a large cardinal axiom. This talk is devoted to our dreams concerning potential braid and knot invariant constructions using Laver tables, and to some real results in this direction, such as a detailed description of 2- and 3-cocycles for the A_n. The rich structure of the latter, as well as spectacular applications of $\mathcal{F}_1$ to Braid Theory, promise interesting topological consequences.

1. A Laver table is...

We start with a formal presentation of the main characters of our story:

Definition 1.1. ➝ A ***shelf*** is a set S endowed with a binary operation $\triangleright$ satisfying the *(left) self-distributivity condition*

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c). \tag{1}$$

- ➝ The free shelf generated by a single element is denoted by $\mathcal{F}_1$.
- ➝ The ***Laver table*** A_n is the unique shelf $(\{1, 2, 3, \ldots, 2^n\}, \triangleright_n)$ satisfying the initial condition

$$a \triangleright_n 1 \equiv a + 1 \mod 2^n. \tag{2}$$

When working modulo N, we will systematically replace the element 0 with N, which is a less conventional representative of the same class. Further, all formulas in A_n will only hold modulo 2^n, which will be often omitted for brevity.

While the first two notions regularly appear (under different names) in Low-Dimensional Topology, Set Theory and Hopf Algebra Theory, the last one is much more exotic. In this preliminary section we will discuss its origin, explain why it is well defined, and present some of its (rather astonishing) properties.

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Laver tables were discovered by Richard Laver ([Lav95]) as a by-product of his study of iterations of *elementary embeddings* in Set Theory. Concretely, for any set S, the set of its self-embeddings $\mathrm{Emb}(S) := \{ f : S \hookrightarrow S \}$ can be endowed the following shelf structure:

$$f \triangleright g = \begin{cases} fgf^{-1} & \text{on the image } \mathrm{Im}(f) \text{ of } f, \\ \mathrm{Id} & \text{on the complement of } \mathrm{Im}(f). \end{cases}$$

Laver took as S a certain limit rank V_λ and supposed it to admit a non-bijective elementary (= preserving all the properties definable in terms of operation $\in$) self-embedding f_0. This is the famous *Axiom I3* in Set Theory, which can be neither proved nor refuted in Zermelo-Fraenkel axiomatic system. Under this assumption, Laver showed that

- f_0 generates a copy of the free shelf $\mathcal{F}_1$ in $\mathrm{Emb}(V_\lambda)$;
- this copy admits finite quotients of size 2^n, which are precisely our A_n;
- the A_n form a projective system whose inverse limit contains a copy of $\mathcal{F}_1$, and can thus be viewed as finite approximations of $\mathcal{F}_1$.

These results are represented in the upper half of Figure 1; the dotted lines stress that everything holds true only modulo the unprovable set-theoretic Axiom I3.

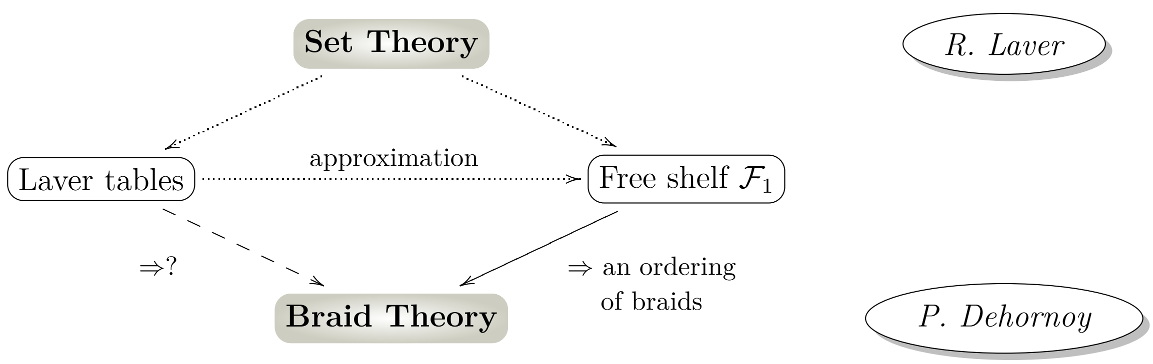

Figure 1: Set-theoretic origins and topological applications of Laver tables

Later, Richard Laver found the completely elementary definition of Laver tables given above. In particular, he proved the following

Theorem 1.2. 1. *For any $n \in \mathbb{N}$, conditions* (1)-(2) *define a unique binary operation on the set* $\{1, 2, 3, \ldots, 2^n\}$.

2. *Laver tables form a projective system of shelves, via the projections*

$$\begin{aligned} p_n \ : \ A_n &\longrightarrow A_{n-1}, \\ a &\longmapsto a \mod 2^{n-1}. \end{aligned}$$

The approximation result mentioned above now appears as

Conjecture 1.3. *The inverse limit of the shelves A_n contains a copy of $\mathcal{F}_1$.*

Much effort has been directed to a proof of this conjecture which would not be based on set-theoretic axioms (see for instance [DJ97, Deh00, Deh14] and references therein), so far without satisfactory results.

A Laver table A_n is presented by its multiplication table, containing the value of $p \triangleright_n q$ in the cell (p, q). One can thus talk about the *columns* and *rows* of A_n. Figure 2 contains the smallest examples. Note that $\triangleright_1$ is nothing else than operation "implication" from Logic (under the identification 1 = False, 2 = True).

Let us now state some combinatorial properties of Laver tables. The last two properties, although elementary-stated, are currently established only under Axiom I3.

- The rows of any Laver table A_n are periodic. Concretely, for every $1 \leqslant p \leqslant 2^n$, there exists an integer 2^r satisfying

$$p + 1 = p \triangleright_n 1 \quad < \quad p \triangleright_n 2 \quad < \quad \cdots \quad < \quad p \triangleright_n 2^r = 2^n,$$

  and the subsequent values $p \triangleright_n q$ then repeat periodically. The number 2^r is called the *period* of p in A_n, and is denoted by $\pi_n(p)$.
- In particular, everyone in the pth row is larger than p, except for the last row.
- Certain rows and columns of A_n are particularly easy to describe:

$$2^n \triangleright_n q = q, \qquad (2^n - 1) \triangleright_n q = 2^n,$$
$$p \triangleright_n 2^n = 2^n, \qquad p \triangleright_n 2^{n-1} = 2^n \text{ if } p \neq 2^n.$$

  The periods of some rows are also easy to determine:

$$\pi_n(2^n) = 2^n, \qquad \pi_n(2^n - 1) = 1,$$
$$\pi_n(2^{n-1}) = 2^{n-1}, \qquad \pi_n(2^n - 2) = \pi_n(2^n - 3) = 2.$$

  However, one does not know any closed formulas either for $p \triangleright_n q$, or for $\pi_n(p)$.
- Any Laver table is generated (as a shelf) by the single element 1. More precisely, A_n is the quotient of the free shelf $\mathcal{F}_1$ (generated by an element 1) by relation

$$(\cdots((1 \triangleright_n 1) \triangleright_n 1) \cdots) \triangleright_n 1 = 1,$$

  where the term 1 is repeated $2^n + 1$ times on the left.
- All other finite monogenerated shelves can be obtained from the A_n by certain canonical procedures described by A. Drápal (cf. [Drá97, Sme13]).
- $\pi_n(1) \underset{n \to \infty}{\longrightarrow} \infty$.
- For all n, one has $\pi_n(1) \leqslant \pi_n(2)$.

All these properties are evidences of the rich combinatorics behind Laver tables.

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

Figure 2: Multiplication tables for the first four Laver tables

2. Dreams: braid and knot invariants based on Laver tables

We now turn to the lower part of Figure 1. We will describe how the arrow on the right works, and how we would like the arrow on the left to work; the dashed line used to draw the latter stresses its partially imaginary character. This section can be seen as motivation for Section 3, and as a presentation of some related open questions.

Shelves have gained recognition among knot theorists due to coloring techniques. Concretely, a *coloring* of a positive braid diagram D by a shelf $(S, \triangleright)$ assigns an element of S to every arc of D in such a way that a b-colored strand becomes $(a \triangleright b)$-colored when it over-crosses an a-colored strand, as shown on Figure 3 A.

Figure 3: Coloring rules for positive and negative crossings

Now, we want colorings to say something about the positive braid β_D represented by D. Therefore, we want Reidemeister III move to induce only local coloring changes, keeping fixed all colors outside the small ball where the move is realized. Figure 4 shows that this happens if and only if operation $\triangleright$ is self-distributive.

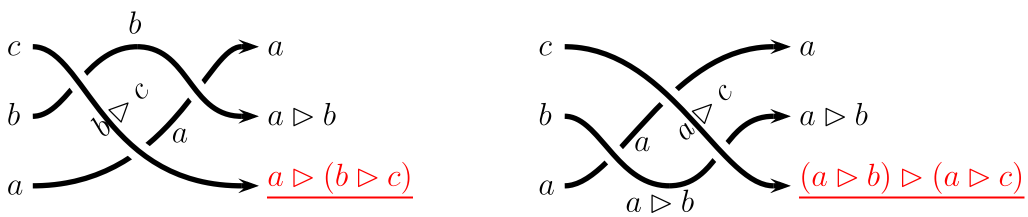

Figure 4: Reidemeister III move $\iff$ self-distributivity

Hence invariants of positive braids can be obtained by counting the number of $(S, \triangleright)$-colorings of their diagrams, or by fixing the colors of all the leftmost arcs and considering the induced colors of the rightmost arcs:

positive braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ shelf

These ideas extends to

- arbitrary braids if $(S, \triangleright)$ is a *rack* — that is, admits a second binary operation $\widetilde{\triangleright}$ with is the left inverse of $\triangleright$, in the sense that

$$a \,\widetilde{\triangleright}\, (a \triangleright b) = b = a \triangleright (a \,\widetilde{\triangleright}\, b); \tag{3}$$

  in this case, the coloring rule from Figure 3 B completes that from Figure 3 A;
- and to knots if $(S, \triangleright)$ is a *quandle* (= a rack where every element is idempotent: $a \triangleright a = a$); only counting invariants are relevant in this case.

Such *shelf/rack/quandle invariants* turn out to be extremely powerful and well adapted for actual calculations.

Laver tables and $\mathcal{F}_1$ are shelves, and thus yield positive braid invariants according to the recipes above. However, they are not racks, except for the trivial A_0. Nevertheless, Patrick Dehornoy managed to refine the above analysis of $\mathcal{F}_1$-colorings and to extract invariants of arbitrary braids out of them (cf. [Deh92, Deh94, Deh00, Kas02]). Let us give some details. To deal with arbitrary braids, one should extend the $(S, \triangleright)$-coloring rule from Figure 3(A) to negative crossings so that Reidemeister II move induces only local coloring changes. For this, the *color propagation map*

$$\begin{aligned} \sigma : S \times S &\longrightarrow S \times S, \\ (a, b) &\longmapsto (a \triangleright b, a) \end{aligned} \tag{4}$$

(see Figure 3(A)) should be invertible, which is equivalent to $(S, \triangleright)$ being a rack. For $\mathcal{F}_1$ the map σ is not surjective but is injective, hence *partially invertible*. Thus one can apply the coloring rule from Figure 3(B) if one has sufficient control on the colors that appear on the left. This control is attained by using a *normal form* for braids, which, roughly, presents a braid as a negative part followed by a positive part, in a way optimal in some sense. Dehornoy showed that for two braids β and β' taken in this normal form, one can always choose some colors $\overline{a} = (a_1, \ldots, a_n)$ on the left which can be propagated all the way to the right along any of the two braids, and that the resulting colors on the right, denoted by $\overline{a}\beta$ and $\overline{a}\beta'$, us the same if and only if $\beta \simeq \beta'$. Moreover, Dehornoy proved that the *left division* relation

$$a \,|_l\, b \qquad \Longleftrightarrow \qquad b = a \triangleright c \text{ for some } c \tag{5}$$

induces a total ordering on $\mathcal{F}_1$, still denoted by $|_l$. For any $k \in \mathbb{N}$, this ordering extends to $\mathcal{F}_1^{\times k}$ in the lexicographical way. Now, relation

$$\beta < \beta' \qquad \Longleftrightarrow \qquad \overline{a}\beta \,|_l\, \overline{a}\beta'$$

turns out to be a well-defined *total left-invariant* (i.e., $\beta < \beta'$ implies $\alpha\beta < \alpha\beta'$) *ordering of braids*. Note that the same ordering can be obtained in a number of ways, algebraic as well as geometric (cf. for instance [FGR$^+$99, SW00, Kas02]). Since its discovery, the braid ordering has been extensively used in the study of braids ([MN03, Mal04, Ito11a, Ito11b, Ito14]). In particular, it is the base of very efficient algorithms for distinguishing braids ([Deh97, Mal01, Dyn03]).

Recall that a Laver table is a quotient of $\mathcal{F}_1$ by (2). This relation destroys the injectivity of the map σ. Since the structure is finite, the map is not surjective either. Therefore Dehornoy's methods do not apply here. However, since at least conjecturally Laver tables are finite approximations of $\mathcal{F}_1$, and the $\mathcal{F}_1$-colorings distinguish all braids, it is natural to expect that A_n-colorings can also say a lot about arbitrary braids. Moreover, because of the finiteness, they are well adapted for computations. The following question thus seems very promising:

Question 2.1. *How can Laver tables be exploited in the investigation of arbitrary braids and knots?*

A deeper understanding of A_n-colorings of positive braids could give a clue to the case of arbitrary braids:

Question 2.2. *What topological or algebraic properties of positive braids can be extracted from A_n-colorings of their diagrams?*

See [Deh14] for an extended discussion of these questions.

3. Reality: 2- and 3-cocycles for Laver tables

In order to simplify the adaptation of A_n-colorings to new contexts — in particular to arbitrary braids, with Question 2.1 in mind — we propose to add more flexibility to their construction. To do this, we use an idea classical to self-distributivity: colorings are enriched with weights. These weights are calculated in a special way using some integer-valued functions on $A_n^{\times 2}$ or $A_n^{\times 3}$, which are in fact 2- and 3-cocycles for the renowned rack cohomology theory for A_n. In [DL14], P. Dehornoy and the author gave a complete description of these cocycles, and showed that they capture all essential combinatorial properties of Laver tables. This section is devoted to details.

We start with recalling the basics of cohomology theory for self-distributive structures, as developed in [FRS95, CJK$^+$03].

Definition 3.1. For a shelf $(S, \triangleright)$, its *rack cohomology* $H^k_{\mathrm{R}}(S)$ is defined as the cohomology of the complex $(\mathrm{Hom}(S^{\times k}, \mathbb{Z}), d^k_{\mathrm{R}})$, where

$$(d^k_{\mathrm{R}} f)(a_1, \dots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \dots, a_{i-1}, a_i \triangleright a_{i+1}, \dots, a_i \triangleright a_{k+1}) \\ - f(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{k+1})\big).$$

The 2-*cocycles* from this theory — that is, maps $\phi : S \times S \to \mathbb{Z}$ satisfying

$$\phi(a \triangleright b, a \triangleright c) + \phi(a, c) = \phi(a, b \triangleright c) + \phi(b, c) \tag{6}$$

— are of particular importance. Evaluate such a 2-cocycle on the colors adjacent to each crossing of an $(S, \triangleright)$-colored positive braid diagram as shown on Figure 5, and sum up the values obtained. The result is called *the (Boltzmann) weight* of the coloring. Figure 5 proves that the multi-set of the weights of all possible $(S, \triangleright)$-colorings is an invariant of positive braids.

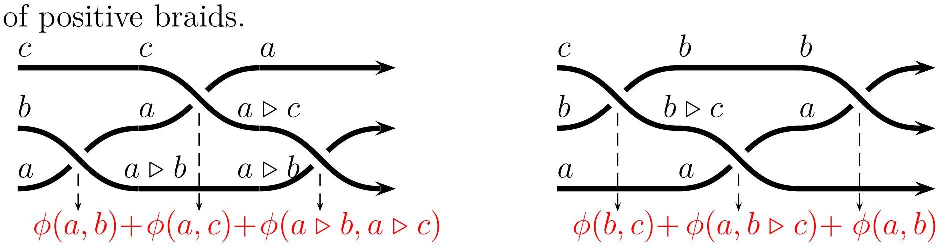

Figure 5: Two-cocycle $\phi \rightsquigarrow$ Boltzmann weights for colored diagrams

These *cocycle invariants* sharpen the shelf invariants obtained by a simple counting of colorings: the latter appear when ϕ is any constant 2-cocycle. A slight modification of this method involves region coloring and rack 3-cocycles; see Figure 6, where region colors are put in boxes, and only relevant colors are indicated for the sake of readability.

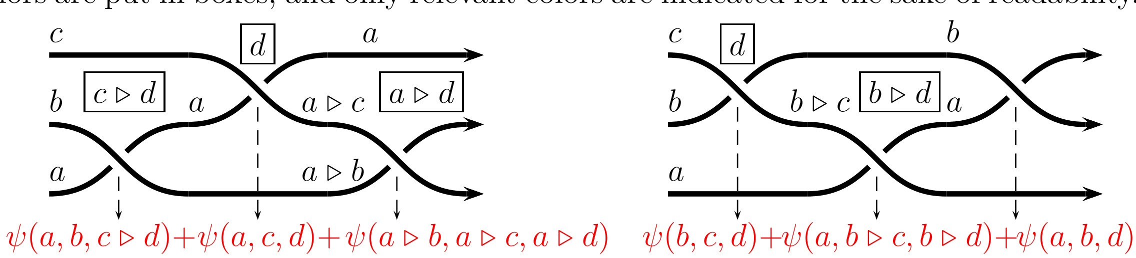

Figure 6: Three-cocycle $\psi \rightsquigarrow$ Boltzmann weights for colored diagrams

One can thus extract a whole system of invariants out of a single shelf:

$$\text{positive braid invariants} \quad \overset{\text{colorings \&}}{\underset{\text{weights}}{\leftrightsquigarrow}} \quad \text{shelf \& 2- or 3-cocycle}$$

In order to feed Laver tables into the machinery above, one should first explicitly calculate their 2- and 3-cocycles. This was done in [DL14]:

Theorem 3.2. *1. For every $n \geqslant 0$, the 2-cocycles for A_n make a free $\mathbb{Z}$-module of rank 2^n, with a basis consisting of the constant cocycle and of $2^n - 1$ explicit $\{0,1\}$-valued coboundaries defined for $1 \leqslant q < 2^n$ by*

$$\phi_{q,n}(a,b) = \begin{cases} 1 & \textit{if } q \textit{ occurs in the column } b\textit{, but not in the column } a \triangleright_n b \textit{ of } A_n, \\ 0 & \textit{otherwise.} \end{cases}$$

2. For every $n \geqslant 0$, the 3-cocycles for A_n make a free $\mathbb{Z}$-module of rank $2^{2n} - 2^n + 1$, with a basis consisting of the constant cocycle and of $2^{2n} - 2^n$ explicit $\{0, \pm 1\}$-valued coboundaries.

The value tables for $\phi_{q,3}$ are presented on Figure 7; the cell (a,b) of such a table contains the value of $\phi_{q,3}(a,b)$, and notation $\cdot$ replaces 0 for a better readability.

$\phi_{1,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	·	·	·	·
2	1	·	·	·	·	·	·	·
3	1	·	·	·	·	·	·	·
4	1	·	·	·	·	·	·	·
5	1	·	·	·	·	·	·	·
6	1	·	·	·	·	·	·	·
7	1	·	·	·	·	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{2,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{3,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{4,3}$	1	2	3	4	5	6	7	8
1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{5,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
5	1	·	·	·	1	·	·	·
6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{6,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{7,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
7	1	·	1	·	1	·	1	·
8	·	·	·	·	·	·	·	·

Figure 7: Two-coboundaries for A_3

It turns out that 2-cocycles capture a lot of combinatorial information about the structure of Laver tables — certainly a promising feature in view of potential applications. We give one example here; see [DL14] for other illustrations.

Proposition 3.3. *For every n, the 2-cocycle $\phi_{2^{n-1},n}$ encodes periods in A_n in the sense that, for every $p < 2^n$, the value of $\pi_n(p)$ is the smallest q satisfying $\phi_{2^{n-1},n}(p,q) = 1$.*

Compare in particular the value table for $\phi_{4,3}$ and the periods for A_3, which can be read from its multiplication table (Figure 2).

We showed that 2- and 3-cocycles for Laver tables yield rich families of positive — and potentially arbitrary — braid invariants. However, a deeper understanding of these invariants is missing. Question 2.2 can thus be upgraded as follows:

Question 3.4. *What topological or algebraic properties of positive braids can be extracted from A_n-colorings of their diagrams, weighted using rack 2- or 3-cocycles?*

4. Bonus: right division ordering for Laver tables

Our description of 2-cocycles for Laver tables (Theorem 3.2) contains an explicit $\{0,1\}$-valued basis. Being $\{0,1\}$-valued is extremely important for combinatorial interpretations. In particular, the Boltzmann weight associated to such a cocycle simply counts crossings colored according to some patterns. A study of these patterns is thus the only ingredient missing for understanding the invariants produced. Now, the construction of this $\{0,1\}$-valued basis in [DL14] heavily used a (quite surprising) new partial ordering on Laver tables, which is also of independent interest. It is discussed in this section.

Recall the left division relation (5), which can be defined for any shelf. As mentioned above, it induces a total ordering on the free shelf $\mathcal{F}_1$. For Laver tables this relation is less interesting, since its transitive closure is the trivial relation: $2^n \triangleright_n q = q$ and $p \triangleright_n 2^n = 2^n$ imply $p \mid_l 2^n \mid_l q$ for all p, q. However, the *right division* relation

$$a \mid_r b \qquad \Longleftrightarrow \qquad b = c \triangleright a \text{ for some } c$$

is much more profound for the A_n, as was shown in [DL14]:

Theorem 4.1. *For a Laver table A_n, consider relation $\mid_r$.*

1. *This relation is a partial ordering.*
2. *This ordering can be alternatively defined as follows:*

$$a \mid_r b \qquad \Longleftrightarrow \qquad Column(a) \supseteq Column(b),$$

   where $Column(x)$ is the set of all elements contained in the xth column of A_n.
3. *The minimal and maximal elements w.r.t. $\mid_r$ are, respectively, 1 and 2^n.*
4. *Any two columns of A_n have different contents.*

Note that a thorough study of the columns of Laver tables was initiated earlier by A. Drápal with a completely different motivation ([Drá95, Drá97]).

Hasse diagrams for the ordering $\mid_r$ on the first Laver tables are presented on Figure 8. In the top two diagrams, each node is accompanied with the content of the corresponding column. One notes that the ordering is linear for $n = 2$, and not linear for $n = 3, 4$ since for instance 2 and 3 are not comparable. For $n \leqslant 4$ one gets lattice orderings, since any two elements admit a least upper bound (and a greatest lower bound); however, this is no longer the case for $n \geqslant 5$.

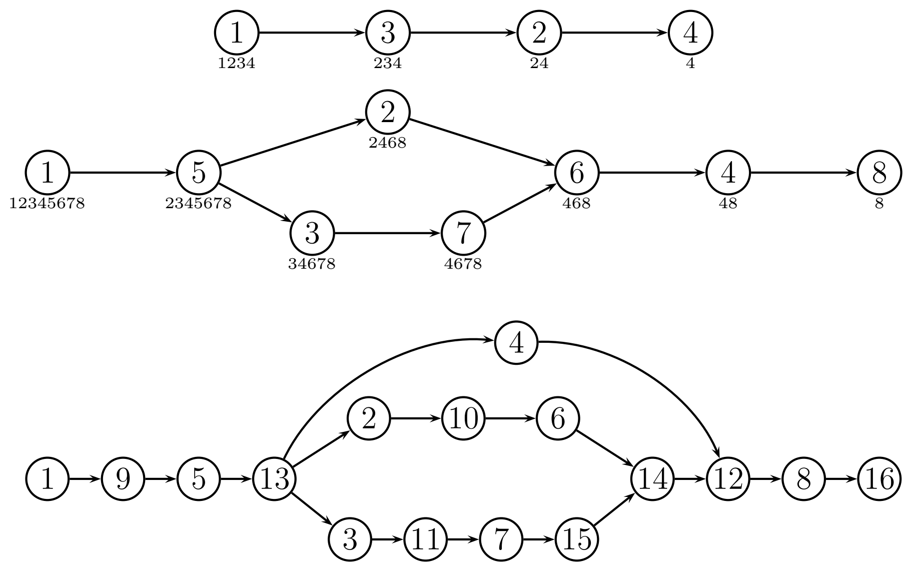

Figure 8: Partial ordering $|_r$ on A_n for $n \leqslant 4$

For completeness, let us discuss the right division relation $|_r$ for $\mathcal{F}_1$. Consider the *depth* function $d : \mathcal{F}_1 \to \mathbb{N}$, recursively defined by $d(g) = 1$, where g is the generator of $\mathcal{F}_1$, and $d(a \triangleright b) = d(b) + 1$ for all a, b. One checks that this function is well-defined. Now, $a \,|_r\, b$ implies $d(b) = d(a) + 1$. Hence relation $|_r$ is not transitive, but it induces a partial ordering on $\mathcal{F}_1$ which in some sense sharpens the depth function. This ordering is not total: for example, the elements $a_k = (\cdots((g \triangleright g) \triangleright g) \cdots) \triangleright g$ with k occurrences of g are pairwise distinct but not distinguishable by d, since $d(a_k) = 2$ for all $k \geqslant 2$. As for now, we are not aware of any applications of this ordering on $\mathcal{F}_1$.

The properties and applications of the two division relations for Laver tables and for $\mathcal{F}_1$ are summarized in Table 1. The most interesting cells are highlighted in grey.

	$a \,\|_r\, b$ if $b = c \triangleright a$	$a \,\|_l\, b$ if $b = a \triangleright c$
A_n	is a partial ordering $\rightsquigarrow$ a good base for 2-cocycles	induces a trivial relation
$\mathcal{F}_1$	induces a partial ordering $\rightsquigarrow$?	induces a total ordering $\rightsquigarrow$ an ordering of braids

Table 1: Different orderings for shelves

5. Dreaming once again: rack cohomology for Laver tables and other shelves

Independently of topological applications, rack cohomology calculations for the A_n are instrumental for a better understanding of their structure. In [DL14], we treated only small degrees. Here we speculate about what one expects in higher degrees.

Theorem 3.2 implies that $H^k_{\mathrm{R}}(A_n) \simeq \mathbb{Z}$ for all n and for $k \leqslant 3$. Preliminary computations confirm that it still holds true for $k = 4$. However, calculation methods for general k are still missing.

Conjecture 5.1. *For all Laver tables A_n and integers k, the rack k-cocycles for A_n form free modules over $\mathbb{Z}$ of rank $\theta_k(2^n)$, where θ_k is a degree $k-1$ polynomial with integer coefficients. Moreover, one has $H^k_{\mathrm{R}}(A_n) \simeq \mathbb{Z}$, with (the equivalence class of) the constant cocycle $f(a_1, \ldots, a_k) = 1$ as generator.*

It would be particularly interesting to find explicit formulas for the polynomials θ_k and to study their properties.

Further, as follows from the work of A. Drápal ([Drá97, Sme13]), all finite shelves with a single generator can be regarded as "interpolations" between Laver tables and *cyclic shelves* C_m (i.e., sets $\{1, 2, 3, \ldots, m\}$ endowed with the operation $a \circ_m b \equiv b + 1 \mod m$). Like for Laver tables, first cohomology groups for the C_m turn out to be isomorphic to $\mathbb{Z}$.

Conjecture 5.2. *For all finite mono-generated shelves S, one has $H^k_{\mathrm{R}}(S) \simeq \mathbb{Z}$.*

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