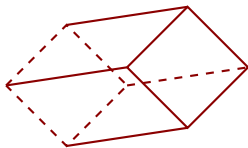# Self-distributive cohomology:
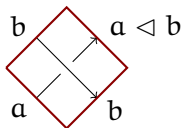# Why care, and how to compute

Victoria LEBED, Trinity College Dublin (Ireland)
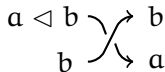
Denver, July 2017

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$$

Knot diagram colorings by $(S, \lhd)$:

$$b \nearrow a \lhd b \qquad a \lhd b \searrow b$$
$$a \searrow b \qquad b \nearrow a$$
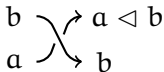
RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
RII	$\forall b, \; a \mapsto a \lhd b$ invertible	rack
RI	$a \lhd a = a$	quandle

Theorem (*Joyce & Matveev* '82):

✓ The number of colorings of a diagram D of a knot K by a quandle $(S, \lhd)$ yields a knot invariant.

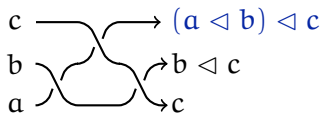✓ $\# \mathrm{Col}_{S, \lhd}(D) = \# \mathrm{Hom}_{\mathrm{Quandle}}(Q(K), S)$

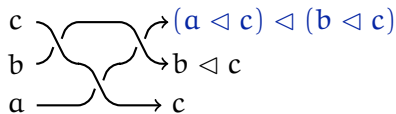where $Q(K)$ = fundamental quandle of K

(a weak universal knot invariant).

Part 1:

You Could Have Invented SD Cohomology If You Were...

diagrams showing RIII move:

left: $c \longrightarrow (a \lhd b) \lhd c$, $b \lhd c$, c

right: $(a \lhd c) \lhd (b \lhd c)$, $b \lhd c$, c

$$\underset{\sim}{\text{RIII}}$$

diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
colorings:	$\mathcal{C}$	$\rightsquigarrow$	$\mathcal{C}'$
coloring sets:	$\mathsf{Col}_{S,\lhd}(D)$	$\overset{1:1}{\longleftrightarrow}$	$\mathsf{Col}_{S,\lhd}(D')$

Counting invariants: $\#\,\mathsf{Col}_{S,\lhd}(D) = \#\,\mathsf{Col}_{S,\lhd}(D')$.

Question: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$

$$\Downarrow$$

$$\big\{\, \omega(\mathcal{C}) \,\big|\, \mathcal{C} \in \mathsf{Col}_{S,\lhd}(D) \,\big\} = \big\{\, \omega(\mathcal{C}') \,\big|\, \mathcal{C}' \in \mathsf{Col}_{S,\lhd}(D') \,\big\}.$$

Answer (*Carter–Jelsovsky–Kamada–Langford–Saito* '03): State-sums over crossings, and Boltzmann weights:

$$\phi \colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum_{b \atop a} \pm \phi(a, b)$$

Conditions on ϕ:

$$\phi(a, b) + \phi(a \lhd b, c) + \cancel{\phi(b, c)} = \cancel{\phi(b, c)} + \phi(a, c) + \phi(a \lhd c, b \lhd c)$$

$$\phi(a, a) = 0$$

Quandle cocycle invariants: $\big\{\, \omega_\phi(\mathcal{C}) \,\big|\, \mathcal{C} \in \mathrm{Col}_{S, \lhd}(D) \,\big\}$.

$$\phi \colon S \times S \to \mathbb{Z}_{\mathfrak{m}} \qquad \rightsquigarrow \qquad \omega_{\phi}(\mathcal{C}) = \sum_{\substack{b \\ a}} \pm \phi(a, b)$$

Quandle cocycle invariants: $\left\{ \, \omega_{\phi}(\mathcal{C}) \, \middle| \, \mathcal{C} \in \mathrm{Col}_{S, \triangleleft}(D) \, \right\}$.

Example: $\phi = 0 \qquad \rightsquigarrow \qquad$ counting invariants.

Quandle cocycle invariants $\supsetneq$ counting invariants.

Conjecture (*Clark–Saito–...*):
    Finite quandle cocycle invariants distinguish all knots.

Generalisation: $K^n \hookrightarrow \mathbb{R}^{n+2}$ and $\phi \colon S^{\times(n+1)} \to \mathbb{Z}_{\mathfrak{m}}$.

Wish:
    $d^{n+1}\phi = 0 \implies \phi$ refines counting invariants for n-knots,
    $\phi = d^n\psi \implies$ the refinement is trivial.

... a Hopf Algebraist

Very open question: Classify nice Hopf algebras over $\mathbb{C}$.
Here "nice" = finite-dimensional pointed.

Applications:
- ✓ cohomology of H-spaces, e.g. Lie groups (*Hopf* '41);
- ✓ invariants of knots and 3-manifolds, TQFT;
- ✓ non-commutative geometry;
- ✓ condensed-matter physics, string theory,

....

Examples:
- ✓ group algebras $\Bbbk G$;
- ✓ enveloping algebras of Lie algebras $U(\mathfrak{g})$;
- ✓ quantum groups: deformations $U_q(\mathfrak{g})$ for semisimple $\mathfrak{g}$,

....

Classification program (*Andruskiewitsch–Graña–Schneider* '98):

nice Hopf algebra A

$\wr$

Yetter–Drinfel'd module $V \in {}^H_H\mathbf{YD}$

$\wr$

braided vector space (V, σ) $\leftsquigarrow$ rack $(S, \lhd)$ & $\phi \colon S \times S \to \mathbb{Z}_{\mathfrak{m}}$

$\wr$

Nichols algebra $B(V)$

$\wr$ & $V \in {}^H_H\mathbf{YD}$

bosonization Hopf algebra $B(V)\#H$

✓ $G(A) =$ the group of group-like elements of A, $H(A) = \mathbb{C}G(A)$;
✓ $R(A) =$ coinvariants of $\mathrm{gr}(A) \twoheadrightarrow \mathrm{gr}(A)_0 = H(A)$, $V(A) = \mathrm{Prim}(R(A))$;
✓ $\sigma \in \mathrm{Aut}(V \otimes V)$, $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$,
                       where $\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$;
✓ in red: "arrows with a large image";
✓ $\mathrm{gr}(A) \cong R(A)\#H(A) = [\text{conjecturally}] = B(V(A))\#H(A)$.

braided vector space $(\mathbb{C}S, \sigma_{\lhd, \phi})$ $\rightsquigarrow$ rack $(S, \lhd)$ & $\phi \colon S \times S \to \mathbb{Z}_m$

$$\sigma_{\lhd, \phi} \colon (a, b) \mapsto q^{\phi(a,b)}(b, a \lhd b)$$

Here q is an mth root of unity, or transcendental.

Wish:

$\quad d^2\phi = 0 \implies (\mathbb{C}S, \sigma_{\lhd, \phi})$ is a braided vector space,

$\quad \phi - \phi' = d^1\psi \implies$ the braided vector spaces are isomorphic.

Rack classification in 3 steps (*Joyce* '82, *Andruskiewitsch–Graña* '03):

1) Simple racks, i.e., without non-trivial quotients:

- ✓ permutation racks $S = \mathbb{Z}_p$, $a \lhd b = a + 1$, p prime;
- ✓ Alexander (= affine) racks $S = \mathbb{Z}_{p^k}$, $a \lhd b = ta + (1-t)b$,
      p prime, t generates $\mathbb{Z}_{p^k}$ over $\mathbb{Z}_p$;
- ✓ certain twisted conjugacy racks: subracks of $(G$, $a \lhd b = f(b^{-1}a)b)$,
      G a group, $f \in \mathrm{Aut}(G)$, $|S|$ divisible by $\geqslant 2$ different primes.

$\wr$ repeated extensions

2) Indecomposable (= connected) racks, i.e., having only 1 orbit w.r.t. $\lhd$.

$\wr$ (various!) glueings

3) General racks.

A rack extension of S is a rack surjection $R \twoheadrightarrow S$.

If S is indecomposable, then $R \cong S \times_\alpha X$, which is $S \times X$ with
$$(a, x) \lhd (b, y) = (a \lhd b, \alpha(a, b, x, y)),$$
where X is a set, and $\alpha \colon S \times S \times X \times X \to X$ satisfies certain axioms.

Important class: abelian rack extensions, i.e., with X an abelian group and
$$\alpha(a, b, x, y) = x + \phi(a, b), \qquad \phi \colon S \times S \to X.$$

Wish:
$$d^2\phi = 0 \implies \phi \text{ defines an abelian extension,}$$
$$\phi = d^1\psi \implies \text{the extension is trivial.}$$

Fenn et al. '95 & *Carter et al.* '03 & *Graña* '00:

Shelf $(S, \lhd)$ & abelian group $X \rightsquigarrow$ cochain complex

$$C_R^k(S, X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$

$$- f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H_R^k(S, X) = \mathrm{Ker}\, d_R^k / \mathrm{Im}\, d_R^{k-1}$.

Quandle $(S, \lhd)$ & abelian group $X \rightsquigarrow$ sub-complex of (C_R^k, d_R^k):

$$C_Q^k(S, X) = \{\, f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \,\}$$

$\rightsquigarrow$ Quandle cohomology $H_Q^k(S, X)$.

$$C_R^k(S, X) = \mathsf{Map}(S^{\times k}, X),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$
$$- f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}));$$

$$C_Q^k(S, X) = \{ f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \}.$$

In small degree:

$(d_R^0 f)(a_1) = 0$

$(d_R^1 f)(a_1, a_2) = f(a_1 \lhd a_2) - f(a_1)$ $\qquad H_R^1(S, X) \cong \mathsf{Map}(\mathsf{Orb}(S), X)$

$(d_R^2 f)(\overline{a}) = f(a_1 \lhd a_2, a_3) - f(a_1, a_3) + f(a_1, a_2) - f(a_1 \lhd a_3, a_2 \lhd a_3)$

$f \in C_Q^2 \iff f(a, a) = 0.$

Remark: $d_R^2 d_R^1 = 0 \iff$ self-distributivity for $\lhd$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing higher knot invariants;
- ✓ an important class of braided vector spaces giving nice Hopf algebras;
- ✓ a parametrization of abelian rack extensions.

$$C_R^k(S, X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1}(f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$
$$- f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}));$$

$$C_Q^k(S, X) = \{\, f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \,\}.$$

✓ For the trivial quandle $T_n = (\{1, \ldots, n\},\ a \triangleleft b = a)$, all $d_R^k = 0$, so
$$H_R^k(T_n, X) \cong X^{n^k}, \qquad\qquad H_Q^k(T_n, X) \cong X^{n(n-1)^{k-1}}.$$

✓ For the free rack on n generators FR_n,
$$H_R^k(FR_n, X) \cong \begin{cases} X, & n = 0, \\ X^n, & n = 1, \\ 0, & n > 1. \end{cases}$$

The quandle cohomology of free quandles has the same form.
(*Fenn–Rourke–Sanderson* '07, *Farinati–Guccione–Guccione* '14)

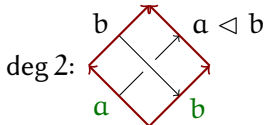# Part 2:

How to Approach SD Cohomology?
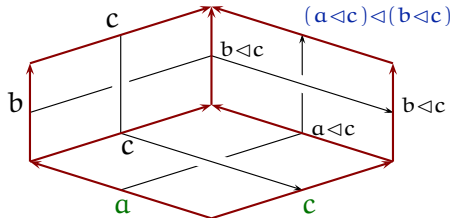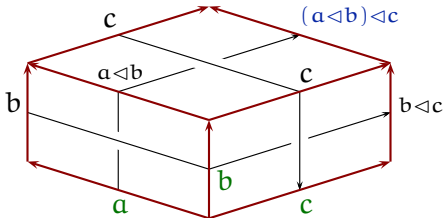
Fenn–Rourke–Sanderson '95:

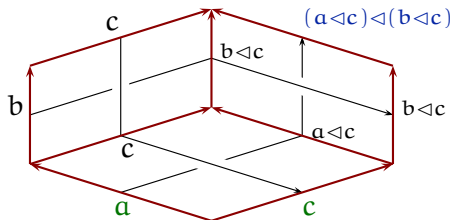Shelf $(S, \lhd)$ $\rightsquigarrow$ rack (= classifying) space $B(S)$. It is a CW-complex:
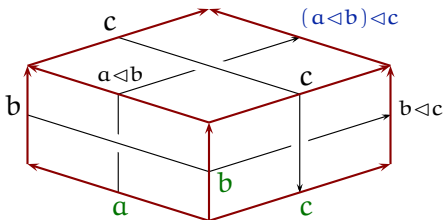
deg 0: $*$ 　　　　　　　　　deg 1: $* \xrightarrow{a} *$

deg 2:
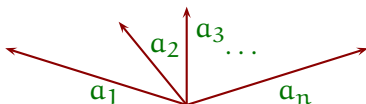

deg 3:

deg 3:

Remark: the edges can be colored starting from the green corner
$\iff$ $\lhd$ is self-distributive.
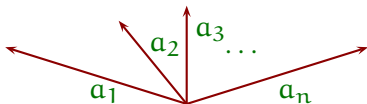
deg n: $\coprod_{S^{\times n}} [0,1]^n$

The coloring continues uniquely to other edges of $[0,1]^n$.

Boundaries: usual topological ones.

$$\boxed{H_R^*(S, X) \cong H^*(B(S), X)}$$

deg n: $\coprod_{S^{\times n}}[0,1]^n$

The coloring continues uniquely to other edges of $[0,1]^n$.

Boundaries: usual topological ones.
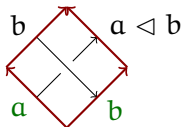
$$H_R^*(S, X) \cong H^*(B(S), X)$$

Nosaka '11: To get quandle cohomology, add 3-dimensional cells bounding

$$H_R^*(S, X) \cong H^*(B(S), X)$$

So, rack spaces bring topological tools in the study of H_R^*.

✓ $\boxed{\pi_1(B(S)) \cong As(S)}$ where $As(S) := \langle S \mid a\, b = b\,(a \lhd b)\rangle$ is the associated (= adjoint = structure = universal enveloping) group of $(S, \lhd)$.

✓ Rack cohomology becomes a pre-cubical cohomology, i.e.,
$$d_R^k = \sum_{i=1}^{k+1} (-1)^{i-1}(d_{i,0}^k - d_{i,1}^k), \qquad d_{i,\varepsilon}d_{j,\zeta} = d_{j-1,\zeta}d_{i,\varepsilon} \quad \text{for all } i < j.$$
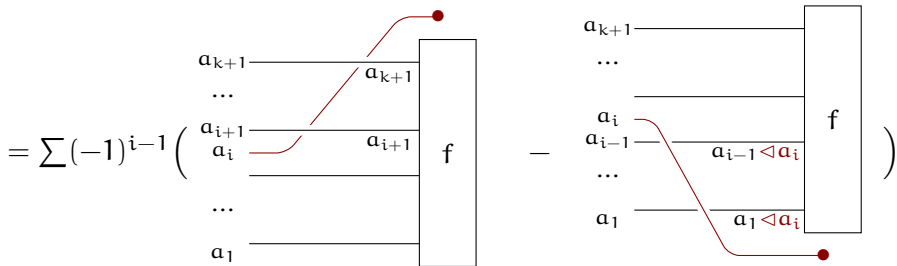
✓ Concrete computations (*Fenn–Rourke–Sanderson* '07):

1) Trivial quandle $T_n = (\{1, \ldots, n\},\ a \lhd b = a)$: $B(T_n) \cong \Omega(\vee_n \mathbb{S}^2)$.

2) Free rack on n generators FR_n: $B(FR_n) \cong \vee_n \mathbb{S}^1$.
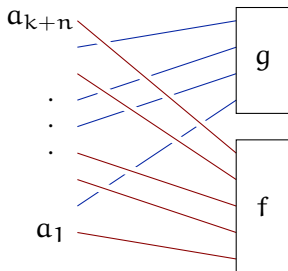
$$C_R^k(S, \mathbb{Z}_m) = \text{Map}(S^{\times k}, \mathbb{Z}_m),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$

$$- f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}))$$

Cup product

$$\smile : C_R^k \otimes C_R^n \to C_R^{k+n}$$

$$f \smile g(a_1, \ldots, a_{k+n}) = \sum_{\text{splittings}} (-1)^{\#\times} \quad \vdots$$

Theorem:

✓ $(C_R^*, \smile)$ is a differential graded associative algebra, graded commutative up to an explicit homotopy;

✓ $(H_R^*, \smile)$ is a graded commutative associative algebra (even better: a dendriform algebra).

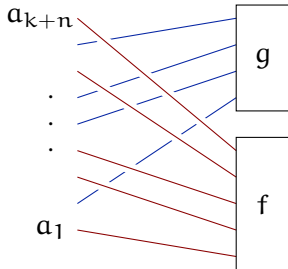$$f \smile g(a_1, \ldots, a_{k+n}) = \sum_{\text{splittings}} (-1)^{\#\times}$$

Theorem:

✓ $(C_R^*, \smile)$ is a differential graded associative algebra, graded commutative up to an explicit homotopy;

✓ $(H_R^*, \smile)$ is a graded commutative associative algebra (even better: a dendriform algebra).

Interpretations:

✓ quantum shuffle coproduct;

✓ topological cup product;

✓ cup product in cubical cohomology.

(*Serre* '51, *Baues* '98, *Clauwens* '11, *Covez* '12, *L.* '17.)

$$C_R^k(S, X) = \text{Map}(S^{\times k}, X),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1})$$
$$- f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$

Theorem (*Etingof–Graña* '03): If $(S, \lhd)$ is a rack and $\# \text{Inn}(S) \in X^*$, then

$$\boxed{H_R^k(S, X) \cong \text{Map}(\text{Orb}(S)^{\times k}, X)} \qquad \text{i.e., } b_k(S) = |\text{Orb}(S)|^k$$

✓ $\text{Orb}(S) = \{$ orbits of S w.r.t. $a \sim a \lhd b \}$;

✓ $\text{Inn}(S)$ is the subgroup of $\text{Aut}(S)$ generated by $t_b \colon a \mapsto a \lhd b$.

Bad news: If $\# \text{Inn}(S) \in X^*$, then

quandle cocycle invariants = coloring invariants + linking numbers.

Hope: Look at $X = \mathbb{Z}_p$, or at the p-torsion of $H_R^k(S, \mathbb{Z})$, where $p \mid \# \text{Inn}(S)$.

Theorem (*Etingof–Graña* '03): If $(S, \lhd)$ is a rack and $\# \operatorname{Inn}(S) \in X^*$, then

$$H_R^k(S, X) \cong \operatorname{Map}(\operatorname{Orb}(S)^{\times k}, X) \qquad \text{i.e., } b_k(S) = |\operatorname{Orb}(S)|^{\times k}$$

✓ $\operatorname{Orb}(S) = \{$ orbits of S w.r.t. $a \sim a \lhd b\}$;

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Hope: Look at $X = \mathbb{Z}_p$, or at the p-torsion of $H_R^k(S, \mathbb{Z})$, where $p \mid \# \operatorname{Inn}(S)$.

It works, and yields interesting invariants! (Wait for Scott's talk).

Theorem (*Dehornoy–L.* '14, *L.* '16): If $(S, \lhd)$ is a finite monogenic shelf (e.g., a Laver table A_n), then $H_R^k(S, X) \cong X$.

Remark: The classes of constant maps do not yield generators in general.

Question: Cohomology of infinite monogenic shelves? Of free shelves?

$$C_R^k(S, X) = \mathrm{Map}(S^{\times k}, X),$$
$$C_Q^k(S, X) = \{\, f\colon S^{\times k} \to X \mid f(\dots, a, a, \dots) = 0 \,\}$$

Theorem (*Litherland–Nelson* '03): The rack cohomology of a quandle splits:

$$H_R^k \cong H_Q^k \oplus H_D^k$$

Here H_D^k is the cohomology of an explicit degenerate subcomplex of C_R^k.

Generalization (*L.–Vendramin* '17): A similar splitting holds for skew cubical cohomology.

Theorem (*Przytycki–Putyra* '16): Degenerate cohomology is degenerate. That is, H_Q^k completely determines H_D^k.

The associated (= adjoint = structure = universal enveloping) group of $(S, \lhd)$:
$$As(S) := \langle S \mid a\,b = b\,(a \lhd b) \rangle$$

Theorem (*Joyce* '82): One has a pair of adjoint functors
$$As : \mathbf{Rack} \rightleftarrows \mathbf{Group} : Conj.$$

Theorem (*Etingof–Graña* '03): $H_R^2(S, X) \cong H_G^1(As(S), Map(S, X))$.

Theorem (*García Iglesias & Vendramin* '16): For a finite indecomposable quandle S,
$$H_R^2(S, X) \cong X \times Hom(N(S), X).$$

Here $N(S)$ is a finite group (the stabilizer of an $a_0 \in S$ in $[As(S), As(S)]$).

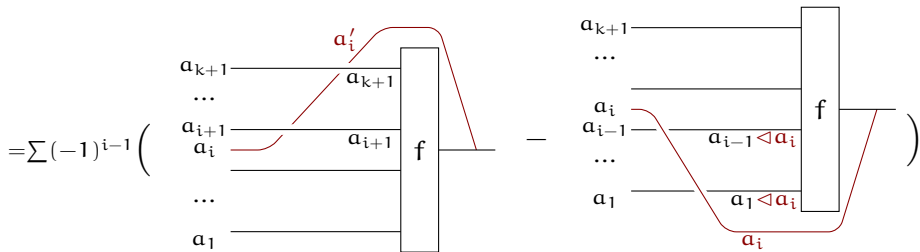**Theorem**. There is a graded algebra morphism $HH^*(As(S), X) \to H_R^*(S, X)$.
Interpretations:

✓ explicit map: quantum symmetrizer (*Covez* '12, *Farinati & García Galofre* '16);

✓ $B(S) \to B(As(S))$ (*Fenn–Rourke–Sanderson* '95).

Level 1: For $M \in {}_{\mathsf{As}(S)}\mathsf{Mod}_{\mathsf{As}(S)}$, the cohomology $H^*_R(S, M)$ is defined by

$$C^k_R(S, M) = \mathrm{Map}(S^{\times k}, M),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) \cdot a'_i$$

$$- a_i \cdot f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1}))$$
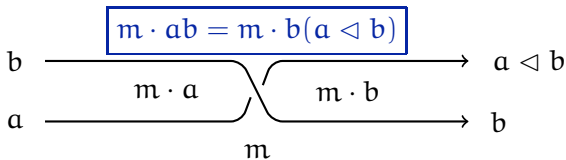

$$a'_i = (\ldots (a_i \lhd a_{i+1}) \ldots) \lhd a_{k+1}.$$

$M \in {}_{As(S)}Mod_{As(S)} \quad \rightsquigarrow \quad H^*_R(S, M).$

Many of the above constructions and results generalize to this setting, e.g. the classifying space:

deg 0: $m \in M$,

deg 1: $m \xrightarrow{a} m \cdot a$.

Application: arc-and-region colorings for knots (*Carter–Kamada–Saito* '01).

$$\boxed{m \cdot ab = m \cdot b(a \lhd b)}$$

Examples of As(S)-(bi)modules:

✓ trivial actions;

✓ $As(S) \in {}_{As(S)}Mod_{As(S)}$;

✓ $S \in Mod_{As(S)}$, with the action induced by $a \cdot b = a \lhd b$;

✓ $Mod_{\mathbb{C}[t^{\pm 1}]} \subset Mod_{As(S)}$, with the action induced by $a \cdot b = ta$.

Level 2: M is a Beck module over S, i.e., an abelian group objects in the category **Rack** $\downarrow$ S (*Andruskiewitsch–Graña* '03, *Jackson* '05):

✓ $\rightsquigarrow$ $H_R^*(S, M)$;

✓ classification of a larger class of rack extensions.

Pursuing the homotopical approach further:

Theorem (*Szymik* '17): Quandle cohomology is a Quillen cohomology.

Applications:

✓ excision isomorphisms;

✓ Mayer–Vietoris exact sequences.