More algebraic connections: SD and the Yang–Baxter equation, Leibniz algebras etc.

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Denver, July 2017

Part 1:

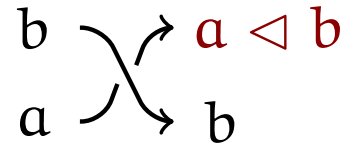
Self-Distributivity and

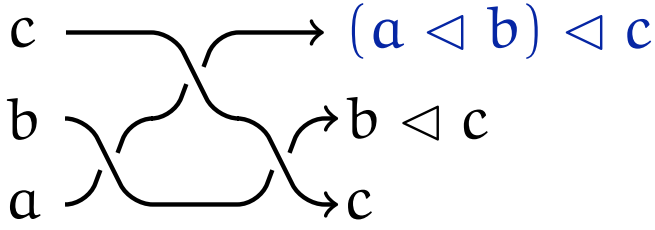
Representations of Braid Groups

Coloring invariants for positive braids

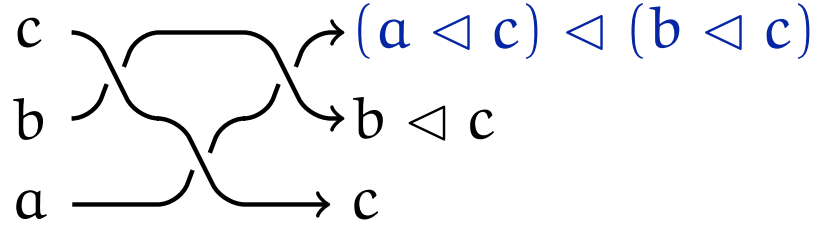
Self-distributivity: $\boxed{(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)}$

Diagram colorings by $(S, \lhd)$
for positive braids:



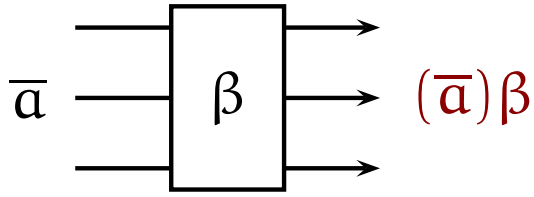


RIII
$\sim$



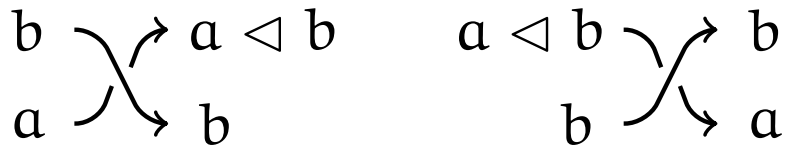
$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$

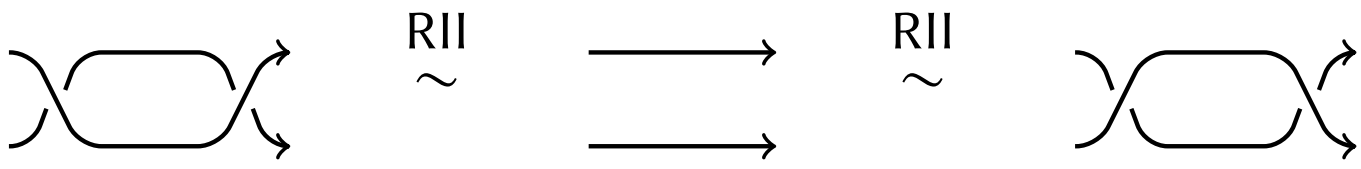
B_n^+ is the monoid of positive braids.



2 Coloring invariants for braids

Diagram colorings by $(S, \triangleleft)$ for braids:





$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$\forall b,\ a \mapsto a \triangleleft b$ invertible	rack
$S \hookrightarrow (S^n)^{B_n}$		$a \triangleleft a = a$	quandle

$a \mapsto (a, \ldots, a)$

B_n is the group of braids.

3 Recovering familiar B_n-representations

$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
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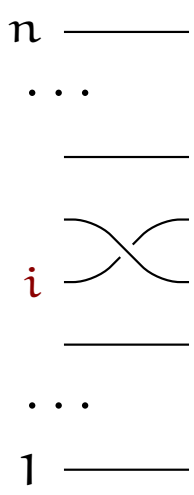
Examples:

S	$a \lhd b$	$(S, \lhd)$ is a	in braid theory
$\mathbb{Z}[t^{\pm 1}]\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$



$$\rho_B(\ ) = I_{i-1} \oplus \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} \oplus I_{n-i-1}$$

Recovering familiar B_n-representations

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$\mathbb{Z}[t^{\pm 1}]\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted linear quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a+1$	rack	$\lg(w), \mathrm{lk}_{i,j}$
free shelf			Dehornoy: order on B_n
Laver tables			???

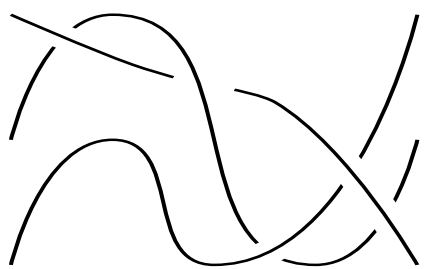
Coloring counting invariants for knots

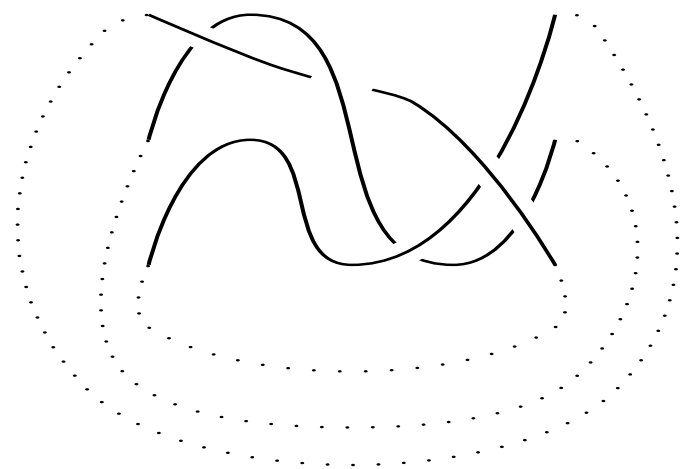
Theorem (*Joyce & Matveev* '82):

✓ The number of colorings of a diagram D of a knot K by a quandle $(S, \triangleleft)$ yields a knot invariant.

✓ $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Hom}_{\mathtt{Quandle}}(Q(K), S) = \mathrm{Tr}(\rho_S(\beta))$

- $Q(K)$ = fundamental quandle of K (a weak universal knot invariant);
- closure$(\beta) = K$;
- $\rho_S \colon B_n \to \mathrm{Aut}(S^n)$ is the S-coloring invariant for braids.



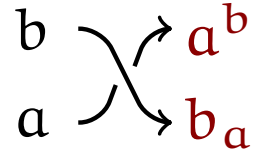


Part 2:

Self-Distributivity and the Yang–Baxter Equation

5 Upper strands matter!

Diagram colorings by (S, σ):



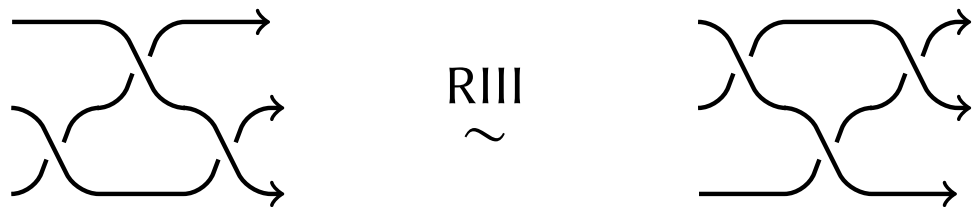
$\sigma(a, b) = (b_a, a^b)$

Ex.: $\sigma_{SD}(a, b) = (b, a \triangleleft b)$

RIII-compatibility $\iff$ set-theoretic Yang–Baxter equation:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon S^{\times 3} \to S^{\times 3}$$

$\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$



In particular,

YBE for σ_{SD} $\iff$ self-distributivity for $\triangleleft$

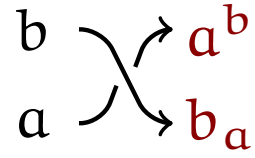
Drinfel′d ’92:

Set-theoretic solutions $\overset{\text{linearize}}{\rightsquigarrow}$ $\overset{\text{deform}}{\rightsquigarrow}$ linear solutions.

Example: $\sigma(a, b) = (b, a)$ $\rightsquigarrow$ R-matrices.

6 YBE as a unifying framework

Diagram colorings by (S, σ):



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$\sigma_1 = \sigma \times \mathrm{Id}_S$, $\sigma_2 = \mathrm{Id}_S \times \sigma$

Exotic example: $\sigma(a, b) = (b, a)$ ⇝

$\sigma_{Lie}(a \otimes b) = b \otimes a + \hbar 1 \otimes [a, b]$, where $[1, a] = [a, 1] = 0$:

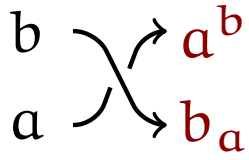
YBE for σ_{Lie} $\iff$ Leibniz relation for $[\,]$

Very exotic example: $\sigma_{Ass}(a, b) = (a * b, 1)$, where $1 * a = a$:

YBE for σ_{Ass} $\iff$ associativity for $*$

YBE and braids and knots

Diagram colorings by (S, σ):



$\sigma(a, b) = (b_a, a^b)$

Ex.: $\sigma_\triangleleft(a, b) = (b, a \triangleleft b)$

RIII	$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$	YB operator
& RII	σ invertible & $\forall b$, $a \mapsto a^b$ and $a \mapsto a_b$ invertible	birack
& RI	$\exists$ a bijection t such that $\sigma(t(a), a) = (t(a), a)$	biquandle

Result: Coloring invariants of braids and knots.

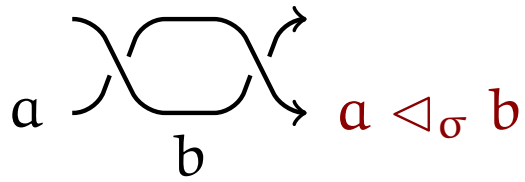
Bad news: These invariants give nothing new!

Unrelated question: Describe free biracks and biquandles.

From biracks to racks

Thm (*Soloviev & Lu–Yan–Zhu* '00, *L.–Vendramin* '17):

✓ Birack (S, σ) $\rightsquigarrow$ its structure rack $(S, \triangleleft_\sigma)$:



✓ This is a projection **Birack** $\twoheadrightarrow$ **Rack** along involutive biracks:

- $\triangleleft_{\sigma_\triangleleft} = \triangleleft$;
- $\triangleleft_\sigma$ trivial $\iff$ $\sigma^2 = \mathrm{Id}$.

✓ The structure rack remembers a lot about the birack:

- $(S, \triangleleft_\sigma)$ quandle $\iff$ (S, σ) biquandle;
- σ and $\triangleleft_\sigma$ induce isomorphic B_n-actions on S^n

$\implies$ same braid and knot invariants.

⚠ $(S, \sigma) \ncong (S, \sigma_{\triangleleft_\sigma})$ as biracks!

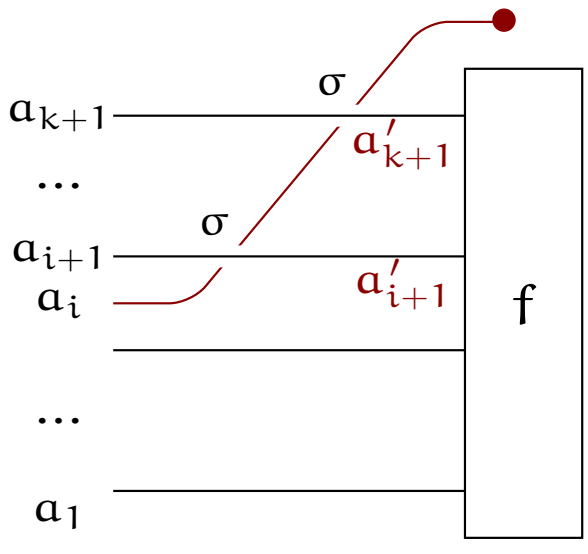
9 Braided cohomology

Carter–Elhamdadi–Saito '04 & *L.* '13:

$$C^k_{Br}(S, \mathbb{Z}_n) = \mathrm{Map}(S^{\times k}, \mathbb{Z}_n),$$

$$(d^k_{Br} f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, a_{i-1}, (a_{i+1}, \ldots, a_{k+1})_{a_i}) - f((a_1, \ldots, a_{i-1})^{a_i}, a_{i+1}, \ldots, a_{k+1}) \big)$$

$$= \sum (-1)^{i-1} \Bigg($$



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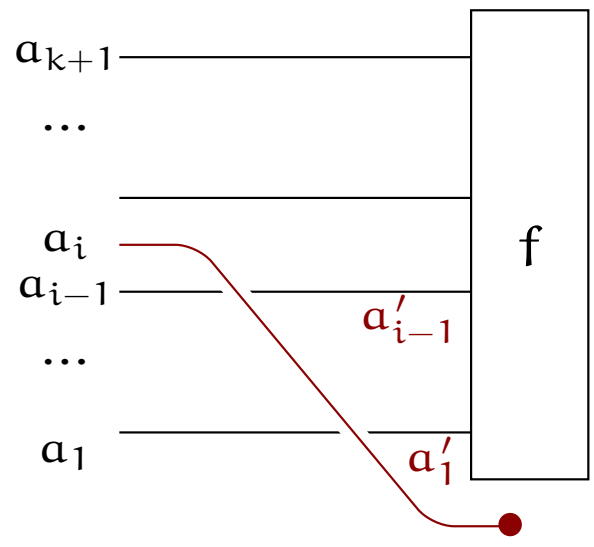


$\Bigg)$

$\rightsquigarrow$ Braided cohomology $H^k_{Br}(S, \mathbb{Z}_n)$.

10 Why I like braided cohomology

① (Higher) braid and knot invariants:

$$d^2_{Br}\phi = 0 \implies \phi \text{ refines (positive) braid coloring invariants,}$$
$$\phi = d^1_{Br}\psi \implies \text{the refinement is trivial.}$$

Question: New invariants?

Answer: I don't know!

② $d^2_{Br}\phi = 0 \implies$ diagonal deformations of σ:

$$\sigma_q(a,b) = q^{\phi(a,b)}\sigma(a,b).$$

(*Freyd–Yetter* '89, *Eisermann* '05)

10 Why I like braided cohomology

③ Unifies cohomology theories for

- ✓ self-distributive structures $\sigma_{SD}(a,b) = (b \lhd a, a)$
- ✓ associative structures $\sigma_{Ass}(a,b) = (a * b, 1)$
- ✓ Lie algebras $\sigma_{Lie}(a \otimes b) = b \otimes a + 1 \otimes [a,b]$

.....

+ explains parallels between them,

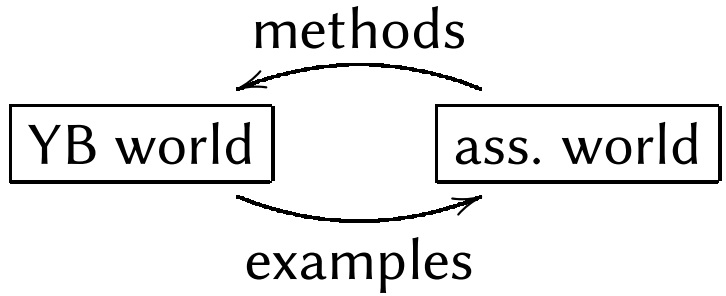
+ suggests theories for new structures.

10 Why I like braided cohomology

④ For certain σ, computes the group cohomology of

$$\boxed{\mathrm{As}(S,\sigma) = \langle\, S \mid ab = b_a\, a^b,\ \text{where } \sigma(a,b) = (b_a\,,\ a^b)\,\rangle}$$

Example: $\mathrm{As}(S,\sigma_{SD}) = \langle\, S \mid a\,b = b\,(a \lhd b)\,\rangle = \mathrm{As}(S,\lhd)$.



Applications: Cohomology of factorized groups & plactic monoids.

Rmk: Structure racks know a lot about structure groups.

Part 3:

Self-Distributivity and

Leibniz Algebras

Leibniz algebras and their cohomology

Bloh '65, *Loday & Cuvier* '91: A Leibniz algebra is a vector space V endowed with a bracket $[,]$ satisfying the Leibniz identity

$$[v,[w,u]] = [[v,w],u] - [[v,u],w].$$

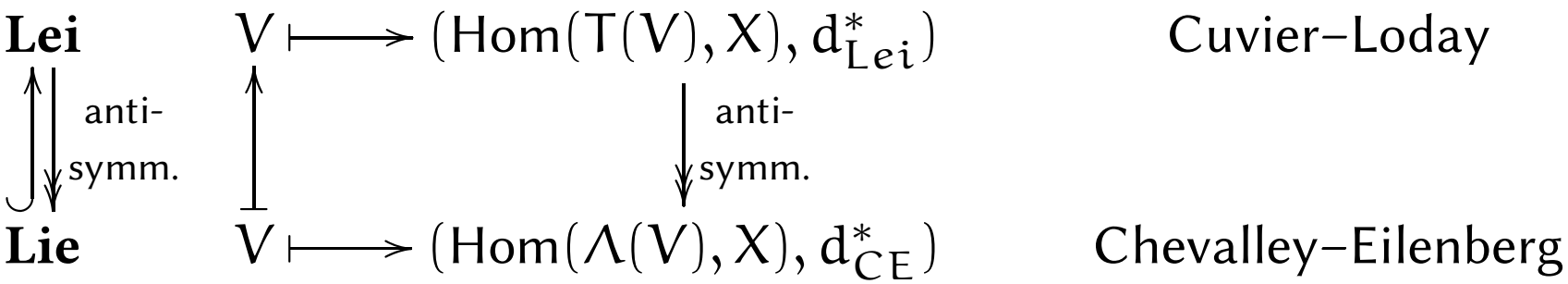
It is a Lie algebra if $[,]$ is antisymmetric: $[v,w] = -[w,v]$.

Leibniz (Loday) cohomology:

$$\begin{array}{lcll} \textbf{Lei} & V \longmapsto & (\mathrm{Hom}(T(V),X), d^*_{Lei}) & \text{Cuvier–Loday} \\ \Big\uparrow\!\!\Big\downarrow \text{anti-symm.} & \Big\uparrow & \Big\downarrow \text{anti-symm.} & \\ \textbf{Lie} & V \longmapsto & (\mathrm{Hom}(\Lambda(V),X), d^*_{CE}) & \text{Chevalley–Eilenberg} \end{array}$$

$$d^{k-1}_{Lei} f(v_1 \dots v_k) = \sum_{1 \leqslant i < j \leqslant k} (-1)^{j-1} f(v_1 \dots v_{i-1} [v_i, v_j] v_{i+1} \dots \widehat{v_j} \dots v_k)$$

Leibniz (Loday) cohomology:



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Remark: This is the braided cohomology of

$$\sigma_{Lie}(a \otimes b) = b \otimes a + 1 \otimes [a, b],$$

where $[1, a] = [a, 1] = 0$. Also, recall that

YBE for σ_{Lie} $\iff$ Leibniz relation for $[\,]$

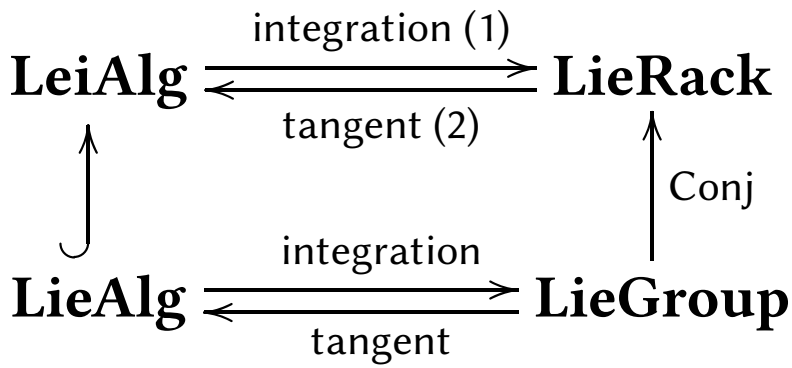
This is one of the explanations of the choice of the Leibniz lift of the Jacobi identity for Lie algebras.

12 Coquecigrue problem

Question (*Loday* '93): $\dfrac{\text{Lie groups}}{\text{Lie algebras}} = \dfrac{???}{\text{Leibniz algebras}}$.

Suggestion (*Kinyon* '07): ??? = Lie rack (= smooth rack).

Criterion 1 Lie's third theorem:



(2) *Kinyon* '07;

(1) *Covez* '10: locally,
Bordemann–Wagemann '16: globally, not functorially.

Coquecigrue problem

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Criterion 2 Cohomological:

(*Loday* '95): A graded algebra morphism, which is iso in degree 1:

$$H^*_{CE}(\mathfrak{g}, X) \to H^*_{Lei}(\mathfrak{g}, X).$$

(*Covez* '12): A graded algebra morphism, injective in degree 1:

$$H^*_G(G, X) \to H^*_R(\mathrm{Conj}(G), X).$$

Part 4:

Self-Distributivity and Cryptography

SD-based authentication scheme

Dehornoy '06: For certain shelves $(S, \lhd)$, it is difficult to reconstruct b from $(a\,,\ a \lhd b)$.

$\rightsquigarrow$ Authentication scheme:

✓ Adam's private key: $s \in S$.

✓ Public key: $(p, p') \in S \times S$, satisfying $p' = p \lhd s$.

✓ Procedure: Adam chooses $r \in S$, and sends to Eve

$$x = p \lhd r,$$
$$x' = p' \lhd r,$$
$$y = s \lhd r.$$

Eve checks $x' = x \lhd y$, i.e.,

$$(p \lhd s) \lhd r = (p \lhd r) \lhd (s \lhd r).$$

15 Multi-distributivity

Multi-shelf = set S + operations $(\lhd_i)_{i\in I}$ satisfying

$$(a \lhd_i b) \lhd_j c = (a \lhd_j c) \lhd_i (b \lhd_j c). \qquad \text{(MD)}$$

Kalka–Teicher '13: SD-based key establishment protocol.

Take $I_A, I_B \subseteq I$ and $S_A, S_B \subseteq S$.

✓ Adam chooses a private key $(a, c, j) \in S \times S_A \times I_A$, and sends to Eve $a \lhd_j c$ and $x_\beta \lhd_j c$ for generators x_β of S_B.

✓ Eve chooses a private key $(b, i) \in S_B \times I_B$, and sends to Adam $x_\alpha \lhd_i b$ for generators x_α of S_A.

✓ Both compute the key (MD).

Suitable types of multi-shelves: S is a group, $f_i, g_i, h_i \in \mathrm{End}(G)$, $a_i \in G$

1) $y \lhd_i x = f_i(x^{-1})g_i(y)h_i(x)$,
2) $y \lhd_i x = xf_i(y)a_i f_i(x^{-1})$.