A journey into the green book

Victoria LEBED, Trinity College Dublin

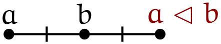
Algebra days 2017, Caen

① An exotic axiom for classical structures

Self-distributivity: $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$

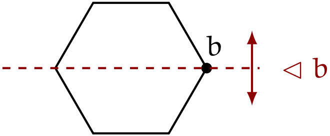
① *Mituhisa Takasaki*, a fresh Japanese maths PhD in 1940 Harbin

Motivation: geometric symmetries.



✓ Abelian group A with $a \lhd b = 2b - a$.

Example: $\mathbb{Z}_n$.

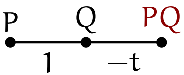


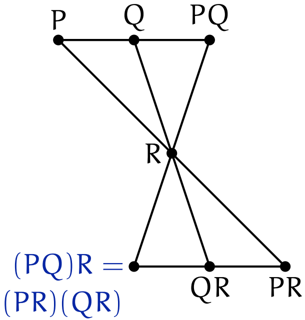
1 An exotic axiom for classical structures

Self-distributivity: $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

② *Gavin Wraith*, a bored school boy

✓ Abelian group A, $t\colon A \to A$, $a \triangleleft b = ta + (1-t)b$.





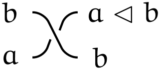
✓ Any group G with $g \triangleleft h = h^{-1}gh$.

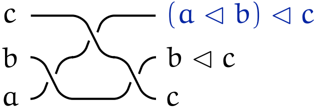
② From curiosity to a theory

Self-distributivity: $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

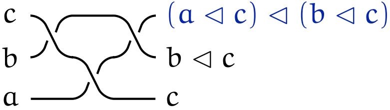
③ *David Joyce & Sergei Matveev*, knot colorists

Diagram colorings by $(S, \triangleleft)$:





$\overset{\text{RIII}}{\sim}$



$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$

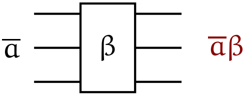
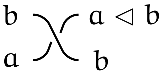


Diagram colorings by $(S, \triangleleft)$



$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$(a \triangleleft b) \mathbin{\tilde{\triangleleft}} b = a = (a \mathbin{\tilde{\triangleleft}} b) \triangleleft b$	rack
$S \hookrightarrow (S^n)^{B_n}$	& RI	$a \triangleleft a = a$	quandle

$a \mapsto (a, \ldots, a)$

Examples:

S	$a \triangleleft b$	$(S, \triangleleft)$ is a	in braid theory
${}_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted linear quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a+1$	rack	$\lg(w), \mathrm{lk}_{i,j}$
free shelf			Dehornoy: order on B_n
Laver table		shelf	???

Progress in Mathematics

Patrick Dehornoy

Braids and Self-Distributivity

Ferran Sunyer i Balaguer
Award winning monograph

Birkhäuser

3 Building bridges

Self-distributivity: $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$

④ *Richard Laver & Patrick Dehornoy*, set theorists hiding from the I3 axiom

✓ Elementary embeddings of certain ranks, with the application operation.

$\Longrightarrow$ Discovery of new shelves from

✓ topology: braids;

✓ algebra:

$\mathcal{F}_1$ = the free shelf on 1 generator γ;

Laver table A_n = $\{ 1, 2, 3, \ldots, 2^n \}$ with the unique (left) SD $\rhd$ satisfying

$$a \rhd 1 \equiv a + 1 \mod 2^n.$$

$$\gamma = 1 \qquad (\gamma \rhd \gamma) \rhd \gamma = 3$$

$$\gamma \rhd \gamma = 2 \qquad ((\gamma \rhd \gamma) \rhd \gamma) \rhd \gamma = 4 \qquad \ldots$$

4 The I3 axiom counter-attacks

Elementary definition: $A_n = (\{1, 2, 3, \ldots, 2^n\}, \triangleright)$ s.t.

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c) \qquad \& \qquad a \triangleright 1 \equiv a + 1 \mod 2^n.$$

Some of the **elementary properties:**

✓ $A_n \cong \mathcal{F}_1 / (\cdots((\gamma \triangleright \gamma) \triangleright \gamma) \cdots) \triangleright \gamma = \gamma$.

✓ $A_n \rightsquigarrow$ all finite monogenic shelves (*Drápal* '97).

✓ Periodic rows.

✓ Solutions of $p \triangleright q = q$.

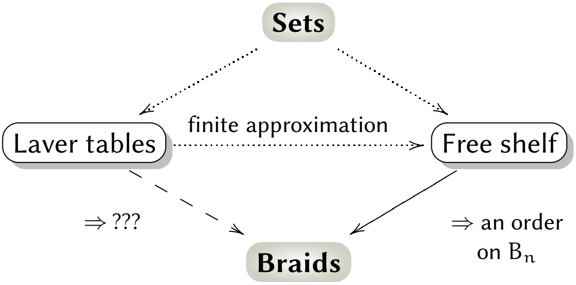
A_3	1	2	3	4	5	6	7	8	
1	2	4	6	8	2	4	6	8	$\pi_3(1) = 4$
2	3	4	7	8	3	4	7	8	$\pi_3(2) = 4$
3	4	8	4	8	4	8	4	8	$\pi_3(3) = 2$
4	5	6	7	8	5	6	7	8	$\pi_3(4) = 4$
5	6	8	6	8	6	8	6	8	$\pi_3(5) = 2$
6	7	8	7	8	7	8	7	8	$\pi_3(6) = 2$
7	8	8	8	8	8	8	8	8	$\pi_3(7) = 1$
8	1	2	3	4	5	6	7	8	$\pi_3(8) = 8$

4 The I3 axiom counter-attacks

Elementary conjectures:

✓ $\pi_n(1) \underset{n\to\infty}{\to} \infty$. ✓ $\pi_n(1) \leqslant \pi_n(2)$. ✓ $\varprojlim_{n\in\mathbb{N}} A_n \supset \mathcal{F}_1$.

Theorems under the axiom I3!



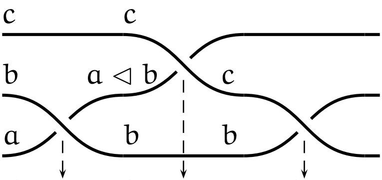
5 Getting more out of colorings

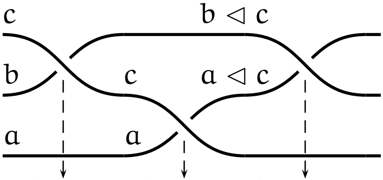
⑤ *Fenn–Rourke–Sanderson & Carter–Jelsovsky–Kamada–Langford–Saito*, refined knot colorists

Shelf S, Abelian group A, $\phi\colon S \times S \to A$ $\rightsquigarrow$ ϕ-weights:

$$S\text{-colored diagram } D \quad \longmapsto \quad \sum \pm\phi(a,b)$$

This is an invariant of colored diagrams iff





$$\phi(a,b)+\phi(a \triangleleft b, c)+\cancel{\phi(b,c)} = \cancel{\phi(b,c)}+\phi(a,c)+\phi(a \triangleleft c, b \triangleleft c)$$

6 New bridges

Shelf $(S, \triangleleft)$ & Ab. group $A \rightsquigarrow (\mathrm{Map}(S^{\times k}, A), d_R^k)$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1}) - f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1})).$$

$\rightsquigarrow$ Rack cohomology $H_R^k(S, A)$

$d_R^2 \phi = 0 \implies \phi$ refines (positive) braid coloring invariants.
$\phi = d_R^1 \psi \implies$ the refinement is trivial.

$(S, \triangleleft)$ quandle, $d_R^k \phi = 0$ & $\cdots \implies$
powerful invariants of $k-1$-dimensional knots in $\mathbb{R}^{k+1}$.

Another **application**: pointed Hopf algebra classification
(*Andruskiewitsch–Graña* '03).

7 Computations

Thm (*Dehornoy–L.* '14):

- $B^2_R(A_n,\mathbb{Z}) \simeq \mathbb{Z}^{2^n-1}$, basis: for $1 \leqslant q < 2^n$,
$$\phi_{q,n}(a,b) = \begin{cases} 1 & \text{if } q \in \mathrm{Column}(b), q \notin \mathrm{Column}(a \triangleright b), \\ 0 & \text{otherwise.} \end{cases}$$
- $B^3_R(A_n) \simeq \mathbb{Z}^{2^{2n}-2^n}$, explicit basis with values in $\{0, \pm 1\}$.
- $H^k_R(A_n) \simeq \mathbb{Z}$, basis: $[f_{const} : \overline{a} \mapsto 1]$ ($k \leqslant 3$).
- $Z^k_R(A_n) \simeq B^k_R(A_n) \oplus H^k_R(A_n)$ ($k \leqslant 3$).

Remark: The $\phi_{q,n}$ capture the combinatorics of A_n (e.g., the periods).

	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	·

	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

	1	2	3	4	5	6	7	8
1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

	1	2	3	4	5	6	7	8
1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
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6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

	1	2	3	4	5	6	7	8
1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
7	1	·	1	·	1	·	1	·
8	·	·	·	·	·	·	·	·

8 More computations

Thm (*L.* '16): For all finite monogenic shelves S,

- $B^k_R(S,\mathbb{Z}) \simeq \mathbb{Z}^{P_k(2^n)}$, $P_k(x) = \dfrac{x^k - x^{k \bmod 2}}{x+1}$.
- $H^k_R(S) \simeq \mathbb{Z}$. ⚠ $[f_{const}]$ is not a basis in general.
- $Z^k_R(S) \simeq B^k_R(A_n) \oplus H^k_R(A_n)$.

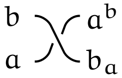
Thm (*Etingof–Graña* '03): For all finite racks S,
$H^k_R(S,\mathbb{Q}) \simeq \mathbb{Q}[f_{const}]$.

Question: Cohomology of $\mathcal{F}_1$?

Prop. (*Farinati–Guccione–Guccione* '14, *Szymik* '17): The cohomology of free racks and quandles is trivial.

9 Getting symmetric

Diagram colorings by (S, σ):



$\sigma(a, b) = (b_a, a^b)$

RIII-compatibility $\iff$ set-theoretic Yang-Baxter equation:

$$\boxed{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon S^{\times 3} \to S^{\times 3}} \qquad \sigma_1 = \sigma \times \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \times \sigma$$

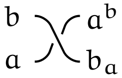
⑥ *Vladimir Drinfel′d*, a divider-and-conqueror

Set-theoretic solutions $\overset{\text{linearise}}{\rightsquigarrow}$ $\overset{\text{deform}}{\rightsquigarrow}$ linear solutions.

Example: $\sigma(x, y) = (y, x)$ $\rightsquigarrow$ R-matrices.

9 Getting symmetric

Diagram colorings by (S, σ):

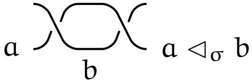


$\sigma(a, b) = (b_a, a^b)$

Question: New coloring invariants of braids?

Thm (*Soloviev & Lu–Yan–Zhu* '00, *L.–Vendramin* '17):

✓ A left non-degenerate set-theoretic YBE solution (S, σ) comes with a shelf operation:



✓ $\lhd_{\sigma_\lhd} = \lhd$.

✓ $\lhd_\sigma$ captures major properties of σ.

✓ Bad news: σ and $\lhd_\sigma$ induce isomorphic B_n^+-actions on S^n.

A better question: New color-and-weight invariants of braids?

Here "weight" = a ϕ-weight for a braided 2-cocycle ϕ.

10 Vote for braided cohomology!

Reasons:

1) $d^2_{\text{Br}}\phi = 0 \implies \phi$ refines (positive) braid coloring invariants.
$\phi = d^1_{\text{Br}}\psi \implies$ the refinement is trivial.
+ invariants of $k-1$-dimensional knots in $\mathbb{R}^{k+1}$
(*Carter–Elhamdadi–Saito* '04).

2) $d^2_{\text{Br}}\phi = 0$, $A = \mathbb{Z}_m \implies$ diagonal deformations of σ:

$$\sigma_q(a,b) = q^{\phi(a,b)}\sigma(a,b)$$

(*Freyd–Yetter* '89, *Eisermann* '05).

3) Unifies cohomology theories for
- ✓ self-distributive structures,
- ✓ associative structures,
- ✓ Lie algebras etc.

+ explains parallels between them,

+ suggests theories for new structures.

Vote for braided cohomology!

4) For certain σ, computes the Hochschild cohomology of

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid ab = b'a' \text{ whenever } \sigma(a,b) = (b',a') \,\rangle$$

methods

braided sets ⇄ groups & algebras

examples

Theorem: $\sigma^2 = \mathrm{Id}$ & … $\Longrightarrow$

- ✓ $\mathrm{Mon}(S,\sigma)$ is of I-type, cancellative, Ore;
- ✓ $\mathrm{Grp}(S,\sigma)$ is solvable, Garside;
- ✓ $\Bbbk\,\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80'-…*).

Question: When is $\mathrm{Grp}(S,\sigma)$ orderable?

Vote for braided cohomology!

4) Braided cohomology computes the Hochschild cohomology of

$$\mathrm{Mon}(S,\sigma) = \langle\, S \mid ab = b'a' \text{ whenever } \sigma(a,b) = (b',a') \,\rangle$$

when:

✓ $\sigma\sigma = \mathrm{Id}$ and $\mathrm{Char}\,\Bbbk = 0$ (*Farinati & García-Galofre* '16);

✓ $\sigma\sigma = \sigma$ (*L.* '16).

Applications:

factorized monoids $G = HK$;

Young tableaux with Schensted multiplication.

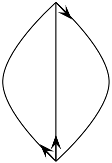
Question: What about the general σ?

5) Comes with a graphical calculus, based on branched braids.

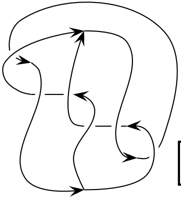
Osaka, Japan

11 Some more topology

Knotted trivalent graphs:



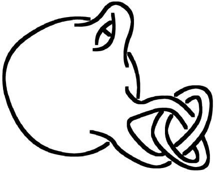
Θ_{st}



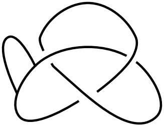
Θ_{KT}

Motivation:

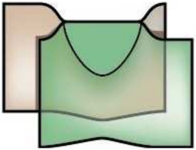
✓ Knotted handle-bodies.



$\longleftrightarrow$



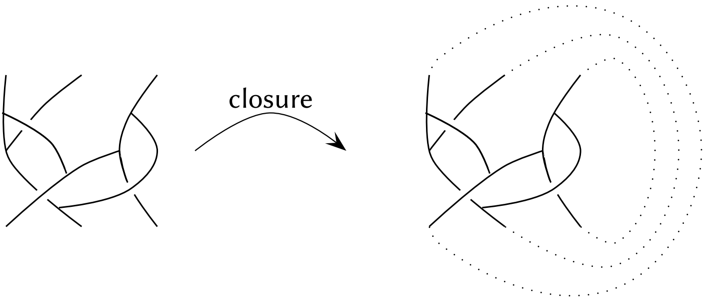
✓ Boundaries of foams.



✓ Form a finitely presented algebraic system (⚠ knots do not).

11 Some more topology

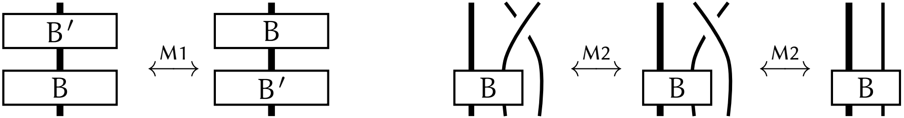
Branched braids | Knotted trivalent graphs



Thm (Kanno–Taniyama ’10, Kamada–L.):

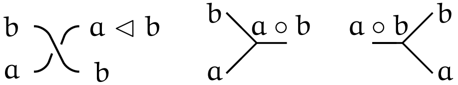
✓ Surjectivity.

✓ Kernel: Markov moves:



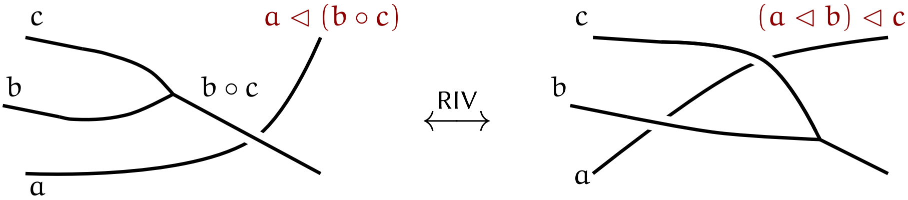
12 A challenge for self-distributivity?

Diagram colorings by $(S, \triangleleft, \circ)$:



Compatible with topology iff

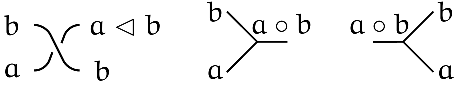
$$(a \circ b) \triangleleft c = (a \triangleleft c) \circ (b \triangleleft c),$$
$$a \triangleleft (b \circ c) = (a \triangleleft b) \triangleleft c,$$
$$a \circ b = b \circ (a \triangleleft b).$$



$\rightsquigarrow$ Powerful color(-and-weight) invariants of branched braids.

13 Laver tables again!

Diagram colorings by $(S, \triangleleft, \circ)$:



Compatible with topology iff

$$(a \circ b) \triangleleft c = (a \triangleleft c) \circ (b \triangleleft c),$$
$$a \triangleleft (b \circ c) = (a \triangleleft b) \triangleleft c,$$
$$a \circ b = b \circ (a \triangleleft b).$$

Thm (*Laver & Drápal* '95): Laver tables with

$$p \circ q = p \triangleright (q+1) - 1$$

satisfy all these axioms.

⚠ False for the free shelf $\mathcal{F}_1$!