# Lecture 9: Computing limits

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# 1 Bricks and mortar

Recall the "bricks and mortar" philosophy for basic functions:

"elementary bricks": the simplest functions

- ✓ $c$, $c \in \mathbb{R}$ (constant functions);
- ✓ $x$;
- ✓ $\sin(x)$;
- ✓ $e^x$ (exponential function)

"mortar": operations

- ✓ arithmetic operations: $f + g$, $f - g$, $fg$, $\frac{f}{g}$;
- ✓ powers: $f^\alpha$, $\alpha \in \mathbb{R}$;
- ✓ composition: $f \circ g$;
- ✓ inverse function $f^{-1}$;
- ✓ gluing functions from different pieces (piecewise defined functions).

# 1 Bricks and mortar

According to this approach, in order lo learn how to compute limits for basic functions, we need to:

1) compute the limits of the functions $c, x, \sin(x), e^x$;

2) understand how limits behave when different operations are performed on functions.

This is our plan for today.

You will be given two theorems, one for each point above. We will not prove them, but it can be easily done starting from the $\varepsilon$-$\delta$ definition of limits (do it as an exercise!).

**Theorem 1.** For any real value $a$, one has

✓ $\lim\limits_{x\to a} c = c$, where $c \in \mathbb{R}$;

✓ $\lim\limits_{x\to a} x = a$;

✓ $\lim\limits_{x\to a} \sin(x) = \sin(a)$.

## 1 Bricks and mortar

**Theorem 2.** Suppose that the two limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist (and are finite). Then

- ✓ $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
- ✓ $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$;
- ✓ $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$;
- ✓ in particular, $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$ for any $c \in \mathbb{R}$;
- ✓ $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (provided that for even $n$, $\lim_{x\to a} f(x) > 0$);
- ✓ $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ whenever $\lim_{x\to a} g(x) \neq 0$.

Further, suppose that $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} h(x) = c$. Then $\lim_{x\to a} h(f(x)) = c$.

⚠ For the operation of taking inverse, things are more complicated: one can have $\lim_{x\to a} f(x) = f(a)$ while $\lim_{x\to f(a)} f^{-1}(x)$ does not exist.

## 2 Limits for polynomial functions

The first class of functions we studied in detail were polynomial functions. Since they are obtained from constant functions and $f(x) = x$ using only addition and multiplication, the two theorems above yield:

$$\lim_{x \to a} P(x) = P(a) \text{ for any polynomial } P \text{ and any real } a.$$

So, to compute the limit of a polynomial function at $a$, you simply evaluate it at $a$.

*Examples.*

$$\lim_{x \to 1} (x^3 - 5x + 1) = 1 - 5 + 1 = -3;$$

$$\lim_{y \to 1} (y^3 - 3y + 1)^5 = (1 - 3 + 1)^5 = -1.$$

# 3 Limits for rational functions

Next, we studied rational functions. They are obtained from polynomial functions using division, and can be written as $\frac{P(x)}{Q(x)}$, where $P$ and $Q$ are polynomials. For rational functions, the two theorems above yield:

$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ for any polynomials $P$ and $Q$ and any real $a$ such that $Q(a) \neq 0$.

Again, to compute such limits, you simply evaluate the function.

With some more work, one proves the following statement:

$\lim_{x\to a^\pm} \frac{P(x)}{Q(x)} = +\infty$ or $-\infty$ for any polynomials $P$ and $Q$ and any real $a$ such that $P(a) \neq 0$, $Q(a) = 0$.

The choice between $+\infty$ and $-\infty$ depends on the signs of $P$ and $Q$ for $x$ close to $a$.

## 3 Limits for rational functions

$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ for any polynomials $P$ and $Q$ and any real $a$ such that $Q(a) \neq 0$.

$\lim_{x \to a^{\pm}} \frac{P(x)}{Q(x)} = +\infty$ or $-\infty$ for any polynomials $P$ and $Q$ and any real $a$ such that $P(a) \neq 0$, $Q(a) = 0$.

*Examples.* $\lim_{x \to 0} \frac{x^2-4x+4}{x^2-4} = \frac{4}{-4} = -1$.

$\lim_{x \to (-2)^+} \frac{x^2-4x+4}{x^2-4} = -\infty$, since $(-2)^2 - 4(-2) + 4 = 16 > 0$, $(-2)^2 - 4 = 0$, and $x^2 - 4 < 0$ for $-2 < x < 0$.

$\lim_{x \to (-2)^-} \frac{x^2-4x+4}{x^2-4} = +\infty$, since $(-2)^2 - 4(-2) + 4 = 16 > 0$, $(-2)^2 - 4 = 0$, and $x^2 - 4 > 0$ for $x < -2$.

The function does not have a finite or infinite two-sided limit at $-2$.

# 4 Indeterminate forms

*Examples.* At point 2, one has

$$\lim_{x\to 2} x^2 - 4x + 4 = \lim_{x\to 2} x^2 - 4 = 0.$$

So, the evaluation approach to $\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4}$ would give $\frac{0}{0}$, which does not make sense!

**Definition.** Fractions $\frac{f(x)}{g(x)}$ where $f(a) = g(a) = 0$ are called **indeterminate forms of type** $0/0$ at $a$.

Limits are particularly useful for studying indeterminate forms. Such forms may have a finite limit, an infinite limit, or no limit at all.

In general, it might be very difficult to describe the behaviour of an indeterminate form. But for the particular case of rational functions, there is an easy algorithm:

## 4 Indeterminate forms

Suppose that $P$ and $Q$ are polynomials, and $P(a) = Q(a) = 0$ for some real $a$. To compute $\lim_{x \to a} \frac{P(x)}{Q(x)}$,

1) Divide both $P$ and $Q$ by $x - a$, to get polynomials $P_2$ and $Q_2$ (this is possible since $P(a) = Q(a) = 0$).

2) If $P_2(a) = Q_2(a) = 0$, repeat Steps 1)-2) for $P_2$ and $Q_2$.

- If $Q_2(a) \neq 0$, then $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P_2(a)}{Q_2(a)}$.

- If $Q_2(a) = 0$, $P_2(a) \neq 0$, then the one-sided limits of $\frac{P(x)}{Q(x)}$ at $a$ are $\pm\infty$, depending on the signs of $P_2(x)$ and $Q_2(x)$ for $x$ close to $a$. The two-sided limit is $\pm\infty$ if the one-sided limits have the same sign; neither finite nor infinite limit exists otherwise.

In Step 1) you can divide by higher powers $(x - a)^k$, if you see a power that will work.

# 4 Indeterminate forms

*Examples.*

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x-2}{x+2}=\frac{2-2}{2+2}=0.$$

$$\lim_{x\to 1}\frac{x^2-4x+3}{x^2-3x+2}=\lim_{x\to 1}\frac{(x-1)(x-3)}{(x-1)(x-2)}=\lim_{x\to 1}\frac{x-3}{x-2}=\frac{1-3}{1-2}=2.$$

$\displaystyle\lim_{x\to 3}\frac{x^2-4x+3}{x^2-6x+9}=\lim_{x\to 3}\frac{(x-1)(x-3)}{(x-3)^2}=\lim_{x\to 3}\frac{x-1}{x-3}$, the limit does not exist.

$$\lim_{x\to 3^+}\frac{x^2-4x+3}{x^2-6x+9}=\lim_{x\to 3^+}\frac{x-1}{x-3}=+\infty,$$

$$\lim_{x\to 3^-}\frac{x^2-4x+3}{x^2-6x+9}=\lim_{x\to 3^-}\frac{x-1}{x-3}=-\infty.$$

So, $\displaystyle\frac{x^2-4x+3}{x^2-6x+9}$ does not have an infinite two-sided limit at $3$ either.

$$\lim_{x\to 0}\frac{x^3-4x^2}{x^4}=\lim_{x\to 0}\frac{x^2(x-4)}{x^4}=\lim_{x\to 0}\frac{x-4}{x^2}=-\infty.$$

# 5 Limits for algebraic functions

Recall that algebraic functions are obtained from constant functions and $f(x) = x$ using addition, multiplication, division, and taking roots of any degree. Their limits can be computed by evaluation if this does not involve division by 0. If division by 0 does occur, you may need a very subtle analysis of the function.

*Examples.*

$$\lim_{x\to 1^-} \frac{x}{\sqrt{1+x}-\sqrt{1-x}} = \frac{1}{\sqrt{1+1}-\sqrt{1-1}} = \frac{1}{\sqrt{2}}.$$

Note that $\lim\limits_{x\to 1^+} \dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}$ does not make sense, since $\sqrt{1-x}$ is not defined for $x > 1$.

## 5 Limits for algebraic functions

$\lim\limits_{x\to 0} \dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}$: here we have an indeterminate form of type $0/0$.

It can be solved by rationalising the denominator:

$$
\begin{aligned}
\lim_{x\to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}} &= \lim_{x\to 0} \frac{x(\sqrt{1+x}+\sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})} \\
&= \lim_{x\to 0} \frac{x(\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)} \\
&= \lim_{x\to 0} \frac{x(\sqrt{1+x}+\sqrt{1-x})}{2x} \\
&= \lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{2} \\
&= \frac{\sqrt{1+0}+\sqrt{1-0}}{2} = 1.
\end{aligned}
$$

Here at all stages we had functions well defined for $x$ close to $0$, so we had the right to perform all the algebraic operations.