You have already encountered limits!

Today we are starting to talk about **limits** and **continuity**, two key notions in this module.

We have already worked with them implicitly in many situations:

- when we plotted the graphs of functions with “holes” (jumps, cuts etc.);
- when we determined **asymptotes** (vertical, horizontal, oblique) for rational functions;
- when we showed that the behaviour of a polynomial
  \[ c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \text{ for large } |x| \]
  is essentially the same as the behaviour of its leading term \( c_n x^n \).
Infinite decimals

You have also been using limits when writing things like

\[ \frac{1}{3} = 0.333333333\ldots. \]

This expression actually means the following:
The numbers 0.3, 0.33, 0.333, 0.3333 etc. get closer and closer to \( \frac{1}{3} \).

Similarly, when writing

\[ \pi = 3.14159265\ldots, \]
we mean that the numbers 3, 3.1, 3.14 etc. approximate the number \( \pi \) better and better.

Limits provide a way of working with approximations in a mathematically rigorous manner.
Informal definition. A function \( f \) is said to have the limit \( L \) as \( x \) approaches \( a \) if the values \( f(x) \) get as close as we wish to \( L \) when \( x \) gets sufficiently close to \( a \). In this case, one writes

\[
\lim_{x \to a} f(x) = L, \quad \text{or} \quad f(x) \to L \quad (x \to a).
\]

The last notation can be read “\( f(x) \) approaches \( L \) when \( x \) approaches \( a \)”.

We say that the limit of a function \( f \) at \( a \) does not exist if there is no \( L \) satisfying \( \lim_{x \to a} f(x) = L \).

⚠️ The definition asks nothing about \( f(a) \), the value of \( f \) at the point \( a \) itself. This point does not even have to lie in the domain of \( f \)!
Example. Let us examine the limit \( \lim_{x \to 1} \frac{x-1}{\sqrt{x-1}} \).

Values for \( x \) getting closer to 1 from the left (i.e., \( x < 1 \)):

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.99</th>
<th>0.999</th>
<th>0.9999</th>
<th>0.99999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1.994987</td>
<td>1.999500</td>
<td>1.999950</td>
<td>1.999995</td>
</tr>
</tbody>
</table>

Values for \( x \) getting closer to 1 from the right (i.e., \( x > 1 \)):

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.00001</th>
<th>1.0001</th>
<th>1.001</th>
<th>1.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2.000005</td>
<td>2.000050</td>
<td>2.000500</td>
<td>2.004988</td>
</tr>
</tbody>
</table>

From this data, it is natural to guess that \( \lim_{x \to 1} \frac{x-1}{\sqrt{x-1}} = 2 \).

But it is only a guess: one has to check that \( f(x) \) approaches 2 for all \( x \) approaching 1, not just for a couple of values of \( x \).
A more systematic approach is to simplify our expression for $f$:

$$\frac{x - 1}{\sqrt{x} - 1} = \frac{(\sqrt{x})^2 - 1}{\sqrt{x} - 1} = \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{\sqrt{x} - 1} = \sqrt{x} + 1$$

for $x \neq 1$. We have the right to write $x = (\sqrt{x})^2$ for $x$ close to 1 (more precisely, for $x \geq 0$).

So, $\frac{x - 1}{\sqrt{x} - 1} = \sqrt{x} + 1$ is close to $\sqrt{1} + 1 = 2$ for $x$ close to 1. (Recall that the value of $f$ at 1 does not matter!)

Conclusion: $\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} = 2$.

In many life situations, this level of rigour is sufficient. However, for certain functions we will need a more precise notion of limit.
Beware of bad samplings!

It is very important that the same qualitative behaviour is observed for all $x$ close to $a$. In the figure below, depending on sampling values of $x$ we may get the limit value

- $1$ as $x$ approaches $0$ following the blue dots;
- $-1$ as $x$ approaches $0$ following the red dots.

Since $1 \neq -1$, the limit of this function at $0$ does not exist.
Limits formally

**Definition.** Suppose that \( f \) is defined on an open interval containing the number \( a \), except possibly for \( x = a \). (That is, on \((b, a) \cup (a, c)\) for some \( b < a < c \).) We say that \( f \) has the limit \( L \) at \( a \), and write \( \lim_{x \to a} f(x) = L \), if for any \( \varepsilon > 0 \), we can find a \( \delta = \delta(\varepsilon) > 0 \) such that
\[
|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.
\]

Observe that \( |f(x) - L| < \varepsilon \) means that \( f(x) \) is in \((L - \varepsilon, L + \varepsilon)\).

Also, \( 0 < |x - a| < \delta \) means that \( x \) is in \((a - \delta, a) \cup (a, a + \delta)\). We emphasize that no information is needed about the value at \( x = a \).

The requirement from the definition can be rewritten as
\[
f(x) \text{ is } \varepsilon\text{-close to } L \iff x \text{ is } \delta\text{-close to } a.
\]

One recognises our earlier informal definition:
\[
f(x) \text{ get as close as we wish to } L \text{ when } x \text{ gets sufficiently close to } a.
\]
Example. Let us prove formally that \( \lim_{x \to 0} x^2 = 0 \) (even though intuitively it is absolutely clear).

We are required to find, given \( \varepsilon > 0 \), a \( \delta > 0 \) so that
\[
|x^2 - 0| < \varepsilon \iff 0 < |x - 0| < \delta.
\]

Discovery phase: If we know that \( |x^2| < \varepsilon \), we can replace \( |x^2| \) by the equal number \( |x|^2 \), and conclude that \( |x|^2 < \varepsilon \), so \( |x| < \sqrt{\varepsilon} \). This suggests that \( \delta(\varepsilon) = \sqrt{\varepsilon} \) should work.

Proof phase: Suppose that we take the value of \( \delta \) we discovered, \( \delta = \sqrt{\varepsilon} \). Let us prove that it fits the purpose we have for it. Assume that \( 0 < |x| < \delta \).

Then
\[
|x^2| = |x|^2 < \delta^2 = (\sqrt{\varepsilon})^2 = \varepsilon,
\]
as required.
Good news

The previous example was there just for your information: proofs like that are frequently done in maths and theoretical physics, but for our purposes it is usually enough to know that things can be made rigorous.

17th century: Newton and Leibniz start the differential and integral calculus

↓ 150 years

19th century: Weierstrass treats limits rigorously, and introduces the $\varepsilon$-$\delta$ machinery.

For our purposes, a good intuitive sense of what having a limit means is usually quite sufficient. We shall formulate a range of theorems about limits to use in applications, but mostly restricting ourselves to intuitive informal proofs, like the one from the “informal” example above.
Karl Weierstraß (1815 – 1897)

\[ f(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \]

\[ ab > 1 + \frac{3}{2} \pi \]

\[ 0 < a < 1 \]

Weierstraß-Funktion

www.marke-individuell.de
One-sided limits

Definition. A function $f$ is said to have the limit $L$ as $x$ approaches $a$ from the right if the values $f(x)$ get as close as we wish to $L$ when $x$ gets sufficiently close to $a$ while staying greater than $a$. In this case, one writes

$$\lim_{x \to a^+} f(x) = L, \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to a^+.$$ 

A function $f$ is said to have the limit $L$ as $x$ approaches $a$ from the left if the values $f(x)$ get as close as we wish to $L$ when $x$ gets sufficiently close to $a$ while staying smaller than $a$. In this case, one writes

$$\lim_{x \to a^-} f(x) = L, \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to a^-.$$ 

Exercise. Give a formal definition of one-sided limits.
One-sided limits

A simple function for which we need one-sided limits is
\[ \text{sign}(x) := \frac{x}{|x|}. \]

Since for all \( x > 0 \) this function assumes the value 1, and for all \( x < 0 \) this function assumes the value \(-1\), we have
\[ \lim_{x \to 0^-} \text{sign}(x) = -1, \quad \lim_{x \to 0^+} \text{sign}(x) = 1. \]
One-sided limits

**Theorem.** Given a function $f$ and a point $a$, one has

$$
\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).
$$

In words, $L$ is the limit of $f$ at $a$ iff $L$ is both its right and its left limit at $a$.

**Exercise.** Prove it.

In the previous example, we had $\lim_{x \to 0^-} \text{sign}(x) = -1$ and $\lim_{x \to 0^+} \text{sign}(x) = 1$. So, the function sign does not have a limit at $0$. 
One-sided limits

Let us consider the following three examples:

In each of those examples, we have
\[ \lim_{x \to -1^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to -1^+} f(x) = 1, \]
so, \[ \lim_{x \to -1} f(x) = 1. \]

This example illustrates the irrelevance of the value at \( x = a \) for the limit \( \lim_{x \to a} f(x) \).
Infinite limits

Definition. A function $f$ is said to have the limit $+\infty$ as $x$ approaches $a$ from the right (left) if the values $f(x)$ increase without bound when $x$ gets sufficiently close to $a$ while staying greater (or smaller) than $a$. In this case, one writes

$$\lim_{x \to a^+} f(x) = +\infty,$$

or

$$f(x) \to +\infty; \quad x \to a^+$$

$$\lim_{x \to a^-} f(x) = +\infty,$$

or

$$f(x) \to +\infty; \quad x \to a^-$$

If both are true, one writes $\lim_{x \to a} f(x) = +\infty$.

Informally, you can think of “increasing without bound” as “getting as close as we wish to $+\infty$.”

⚠️ However, avoid treating $\pm \infty$ as a real number. For instance, the expression $|f(x) - (+\infty)|$ doesn’t make sense!
Infinite limits

Similarly, \( f \) is said to have the limit \(-\infty\) as \( x \) approaches \( a \) from the right (left) if the values \( f(x) \) decrease without bound when \( x \) gets sufficiently close to \( a \) while staying greater (or smaller) than \( a \). One writes

\[
\lim_{x \to a^+} f(x) = -\infty, \quad \text{or} \quad f(x) \to -\infty; \\
\lim_{x \to a^-} f(x) = -\infty, \quad \text{or} \quad f(x) \to -\infty.
\]

If both are true, one writes \( \lim_{x \to a} f(x) = -\infty \).

Exercise. Give a formal definition of infinite limits.

Whenever one of the conditions \( \lim_{x \to a^\pm} f(x) = \pm \infty \) holds, the function \( f \) has a vertical asymptote at \( a \).

⚠️ When \( \lim_{x \to a} f(x) = \pm \infty \), some books say “the limit of \( f \) at \( a \) exists and is infinite”, while other books say “the limit of \( f \) at \( a \) does not exist, but \( f \) has an infinite limit at \( a \)”. We will stick to the second convention.
Infinite limits

Example. Let us consider the function \( f(x) = \frac{1}{x} \):

In this case, we have
\[
\lim_{x \to 0^+} f(x) = +\infty, \quad \lim_{x \to 0^-} f(x) = -\infty.
\]

So, one-sided limits at 0 are infinite, but neither finite nor infinite (two-sided) limit exists at 0.
Example. Let us consider the function \( f(x) = \frac{1}{x^2} \):

In this case, we have \( \lim_{x \to 0^+} f(x) = +\infty = \lim_{x \to 0^-} f(x) \).

So, one-sided limits are infinite, as well as the (two-sided) limit:

\[
\lim_{x \to 0} f(x) = +\infty.
\]