Lecture 8: The notion of limit

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MA1S11A: Calculus with Applications for Scientists

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1 You have already encountered limits!

Today we are starting to talk about **limits** and **continuity**, two key notions in this module.

We have already worked with them implicitly in many situations:

- ✓ when we plotted the graphs of functions with “holes” (jumps, cuts etc.);
- ✓ when we determined asymptotes (vertical, horizontal, oblique) for rational functions;
- ✓ when we showed that the behaviour of a polynomial $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ for large $|x|$ is essentially the same as the behaviour of its leading term $c_n x^n$.

2 Infinite decimals

You have also been using limits when writing things like

$$\frac{1}{3} = 0.333333333\ldots.$$

This expression actually means the following:
The numbers $0.3,\ 0.33,\ 0.333,\ 0.3333$ etc. get closer and closer to $\frac{1}{3}$.

Similarly, when writing

$$\pi = 3.14159265\ldots,$$

we mean that the numbers $3,\ 3.1,\ 3.14$ etc. approximate the number π better and better.

Limits provide a way of working with approximations in a mathematically rigorous manner.

3 Limits informally

Informal definition. A function f is said to **have the limit L as x approaches a** if the values $f(x)$ get as close as we wish to L when x gets sufficiently close to a. In this case, one writes

$$\lim_{x \to a} f(x) = L, \qquad \text{or} \qquad f(x) \underset{x \to a}{\longrightarrow} L.$$

The last notation can be read "$f(x)$ approaches L when x approaches a".

We say that **the limit of a function f at a does not exist** if there is no L satisfying $\lim_{x \to a} f(x) = L$.

⚠ The definition asks nothing about $f(a)$, the value of f at the point a itself. This point does not even have to lie in the domain of f!

3 Limits informally

Example. Let us examine the limit $\lim\limits_{x \to 1} \frac{x-1}{\sqrt{x}-1}$.

Values for x getting closer to 1 from the left (i.e., $x < 1$):

x	0.99	0.999	0.9999	0.99999
$f(x)$	1.994987	1.999500	1.999950	1.999995

Values for x getting closer to 1 from the right (i.e., $x > 1$):

x	1.00001	1.0001	1.001	1.01
$f(x)$	2.000005	2.000050	2.000500	2.004988

From this data, it is natural to **guess** that $\lim\limits_{x \to 1} \frac{x-1}{\sqrt{x}-1} = 2$.

But it is only a guess: one has to check that $f(x)$ approaches 2 for **all** x approaching 1, not just for a couple of values of x.

3 Limits informally

A more systematic approach is to simplify our expression for f:

$$\frac{x-1}{\sqrt{x}-1} = \frac{(\sqrt{x})^2-1}{\sqrt{x}-1} = \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \sqrt{x}+1$$

for $x \neq 1$. We have the right to write $x = (\sqrt{x})^2$ for x close to 1 (more precisely, for $x \geqslant 0$).

So, $\frac{x-1}{\sqrt{x}-1} = \sqrt{x}+1$ is close to $\sqrt{1}+1 = 2$ for x close to 1. (Recall that the value of f at 1 does not matter!)

Conclusion: $\lim\limits_{x \to 1} \frac{x-1}{\sqrt{x}-1} = 2$.

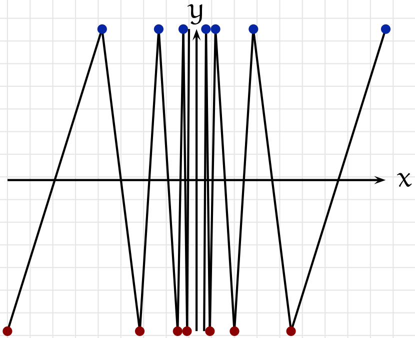
In many life situations, this level of rigour is sufficient. However, for certain functions we will need a more precise notion of limit.

4 Beware of bad samplings!

⚠ It is very important that the same qualitative behaviour is observed for **all** x close to a. In the figure below, depending on sampling values of x we may get the limit value

- ✓ 1 as x approaches 0 following the blue dots;
- ✓ -1 as x approaches 0 following the red dots.

Since $1 \neq -1$, the limit of this function at 0 does not exist.



5 Limits formally

Definition. Suppose that f is defined on an open interval containing the number a, except possibly for $x = a$. (That is, on $(b, a) \cup (a, c)$ for some $b < a < c$.) We say that f **has the limit** L **at** a, and write $\lim_{x \to a} f(x) = L$, if for any $\varepsilon > 0$, we can find a $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Observe that $|f(x) - L| < \varepsilon$ means that $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$.

Also, $0 < |x - a| < \delta$ means that x is in $(a - \delta, a) \cup (a, a + \delta)$. We emphasize that no information is needed about the value at $x = a$.

The requirement from the definition can be rewritten as

$$f(x) \text{ is } \varepsilon\text{-close to } L \quad \Longleftarrow \quad x \text{ is } \delta\text{-close to } a.$$

One recognises our earlier informal definition:

$f(x)$ get as close as we wish to L when x gets sufficiently close to a.

5 Limits formally

Example. Let us prove formally that $\lim_{x\to 0} x^2 = 0$ (even though intuitively it is absolutely clear).

We are required to find, given $\varepsilon > 0$, a $\delta > 0$ so that

$$|x^2 - 0| < \varepsilon \quad \Longleftarrow \quad 0 < |x - 0| < \delta.$$

Discovery phase: If we know that $|x^2| < \varepsilon$, we can replace $|x^2|$ by the equal number $|x|^2$, and conclude that $|x|^2 < \varepsilon$, so $|x| < \sqrt{\varepsilon}$. This suggests that $\delta(\varepsilon) = \sqrt{\varepsilon}$ should work.

Proof phase: Suppose that we take the value of δ we discovered, $\delta = \sqrt{\varepsilon}$. Let us prove that it fits the purpose we have for it. Assume that $0 < |x| < \delta$. Then

$$|x^2| = |x|^2 < \delta^2 = (\sqrt{\varepsilon})^2 = \varepsilon,$$

as required.

6 Good news

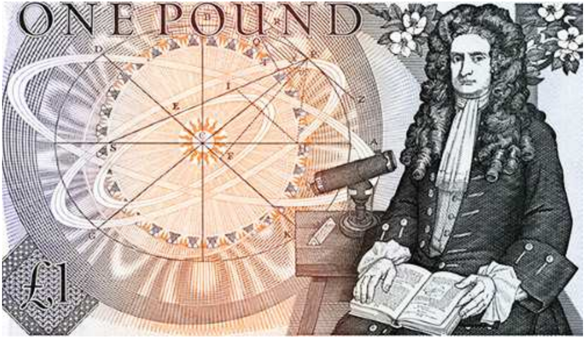
The previous example was there just for your information: proofs like that are frequently done in maths and theoretical physics, but for our purposes it is usually enough to know that things can be made rigorous.

17th century: Newton and Leibniz start the differential and integral calculus

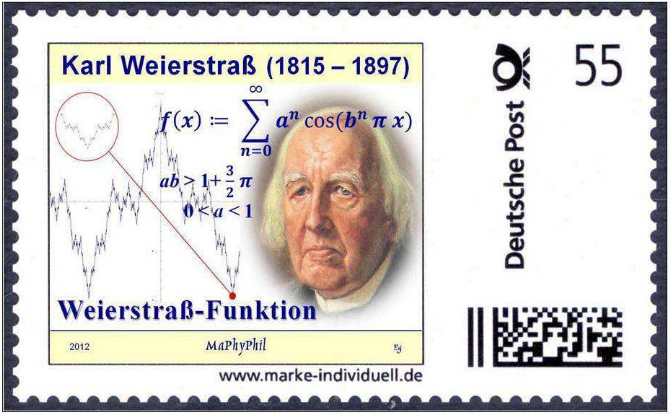
$\downarrow$ 150 years

19th century: Weierstrass treats limits rigorously, and introduces the ε-δ machinery.

For our purposes, a good intuitive sense of what having a limit means is usually quite sufficient. We shall formulate a range of theorems about limits to use in applications, but mostly restricting ourselves to intuitive informal proofs, like the one from the “informal” example above.


ONE POUND
£1


100
GOTTFRIED WILHELM LEIBNIZ 1646-1716 DEUTSCHLAND


Karl Weierstraß (1815 – 1897)
$f(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$
$ab > 1 + \frac{3}{2}\pi$
$0 < a < 1$
Weierstraß-Funktion
2012
MaPhyPhil
Deutsche Post
55
www.marke-individuell.de

7 One-sided limits

Definition. A function f is said to **have the limit L as x approaches a from the right** if the values $f(x)$ get as close as we wish to L when x gets sufficiently close to a while staying greater than a. In this case, one writes

$$\lim_{x \to a^+} f(x) = L, \qquad \text{or} \qquad f(x) \underset{x \to a^+}{\longrightarrow} L.$$

A function f is said to **have the limit L as x approaches a from the left** if the values $f(x)$ get as close as we wish to L when x gets sufficiently close to a while staying smaller than a. In this case, one writes

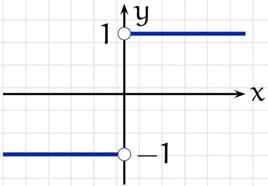
$$\lim_{x \to a^-} f(x) = L, \qquad \text{or} \qquad f(x) \underset{x \to a^-}{\longrightarrow} L.$$

Exercise. Give a formal definition of one-sided limits.

7 One-sided limits

A simple function for which we need one-sided limits is

$$\operatorname{sign}(x) := \frac{x}{|x|}.$$



Since for all $x > 0$ this function assumes the value 1, and for all $x < 0$ this function assumes the value -1, we have

$$\lim_{x \to 0^-} \operatorname{sign}(x) = -1, \qquad \lim_{x \to 0^+} \operatorname{sign}(x) = 1.$$

7 One-sided limits

Theorem. Given a function f and a point a, one has

$$\lim_{x \to a} f(x) = L \qquad \Longleftrightarrow \qquad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

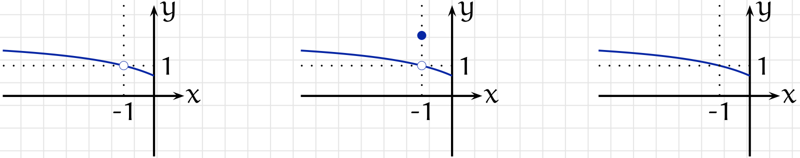
In words, L is the limit of f at a iff L is both its right and its left limit at a.

Exercise. Prove it.

In the previous example, we had $\lim_{x \to 0^-} \operatorname{sign}(x) = -1$ and $\lim_{x \to 0^+} \operatorname{sign}(x) = 1$. So, the function sign does not have a limit at 0.

7 One-sided limits

Let us consider the following three examples:



In each of those examples, we have

$$\lim_{x \to -1^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to -1^+} f(x) = 1,$$

so, $\lim_{x \to -1} f(x) = 1$.

This example illustrates the irrelevance of the value at $x = a$ for the limit $\lim_{x \to a} f(x)$.

8 Infinite limits

Definition. A function f is said to **have the limit $+\infty$ as x approaches a from the right (left)** if the values $f(x)$ increase without bound when x gets sufficiently close to a while staying greater (or smaller) than a. In this case, one writes

$$\lim_{x\to a^+} f(x) = +\infty, \qquad \text{or} \qquad f(x) \underset{x\to a^+}{\longrightarrow} +\infty;$$

$$\lim_{x\to a^-} f(x) = +\infty, \qquad \text{or} \qquad f(x) \underset{x\to a^-}{\longrightarrow} +\infty.$$

If both are true, one writes $\lim_{x\to a} f(x) = +\infty$.

Informally, you can think of "increasing without bound" as "getting as close as we wish to $+\infty$".

⚠ However, avoid treating $\pm\infty$ as a real number. For instance, the expression $|f(x) - (+\infty)|$ doesn't make sense!

8 Infinite limits

Similarly, f is said to **have the limit $-\infty$ as x approaches a from the right (left)** if the values $f(x)$ decrease without bound when x gets sufficiently close to a while staying greater (or smaller) than a. One writes

$$\lim_{x\to a^+} f(x) = -\infty, \qquad \text{or} \qquad f(x) \underset{x\to a^+}{\longrightarrow} -\infty;$$
$$\lim_{x\to a^-} f(x) = -\infty, \qquad \text{or} \qquad f(x) \underset{x\to a^-}{\longrightarrow} -\infty.$$

If both are true, one writes $\lim\limits_{x\to a} f(x) = -\infty$.

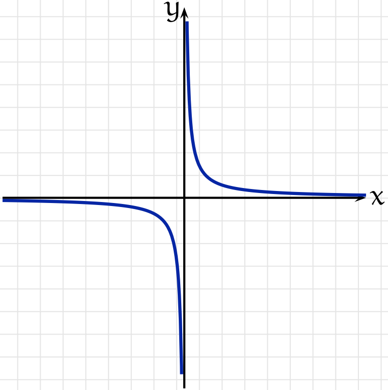
Exercise. Give a formal definition of infinite limits.

Whenever one of the conditions $\lim\limits_{x\to a^\pm} f(x) = \pm\infty$ holds, the function f has a vertical asymptote at a.

⚠ When $\lim\limits_{x\to a} f(x) = \pm\infty$, some books say "the limit of f at a exists and is infinite", while other books say "the limit of f at a does not exist, but f has an infinite limit at a". We will stick to the second convention.

8 Infinite limits

Example. Let us consider the function $f(x) = 1/x$:

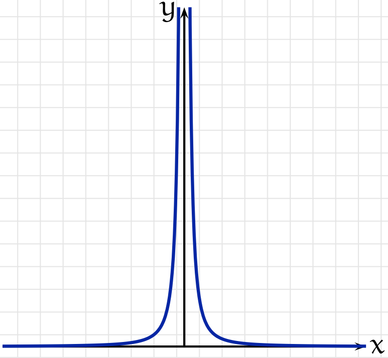


In this case, we have $\lim_{x\to 0^+} f(x) = +\infty$, $\lim_{x\to 0^-} f(x) = -\infty$.

So, one-sided limits at 0 are infinite, but neither finite nor infinite (two-sided) limit exists at 0.

8 Infinite limits

Example. Let us consider the function $f(x) = 1/x^2$:



In this case, we have $\lim_{x\to 0^+} f(x) = +\infty = \lim_{x\to 0^-} f(x)$.

So, one-sided limits are infinite, as well as the (two-sided) limit:

$\lim\limits_{x\to 0} f(x) = +\infty$.