7. Representations of abelian groups

We'll start with the simplest example of abelian groups: **cyclic groups** C_n (also denoted by $\mathbb{Z}_n$), $n \in \mathbb{N}$.

They can be defined in 3 equivalent ways (cf. Lecture 1):

(a) $C_n = \{0, 1, \dots, n-1\}$, with the usual addition operation modulo n.

(b) $C_n = \langle t \mid t^n = 1 \rangle$

(c) $C_n =$ the group of orientation-preserving symmetries of the regular n-gon.

$$(\{0,1,\dots,n-1\}, +) \simeq \langle t \mid t^n = 1 \rangle \simeq \underset{\text{or.-preserving}}{\mathrm{Symm}^+}(⬡)$$

$1 \longleftrightarrow t \longleftrightarrow$ rotation by $\frac{2\pi}{n}$: 

$0 \longleftrightarrow 1 \longleftrightarrow$ identity

⚠ The neutral element is denoted differently in different frameworks.

C_n is abelian $\Rightarrow \forall a, b \in C_n,\ aba^{-1} = b \Rightarrow \forall e \in \mathrm{Conj}(C_n),\ \#e = 1$.

So $\#\mathrm{Irrep}(C_n) = \#\mathrm{Conj}(C_n) = \#C_n = n$, and $\mathrm{Irrep}(C_n) = \{V_0, V_1, \dots, V_{n-1}\}$.

$$\sum_{i=0}^{n-1} \underbrace{(\dim_{\mathbb{C}}(V_i))^2}_{\geq 1} = \#C_n = n \Rightarrow \forall i,\ \dim_{\mathbb{C}}(V_i) = 1.$$

Conclusion: All irreps of C_n are of degree 1

Let us find them.

We are looking for $\rho_j : C_n \xrightarrow[\text{grp}]{} \mathbb{C}^*$.

$t^n = 1 \Rightarrow \underline{\omega_j := \rho_j(t)}$ should satisfy $\omega_j^n = 1$.

Put $\underline{\omega = e^{\frac{2\pi i}{n}}} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$ (a primitive n-th root of 1).

$\omega_j^n = 1 \Leftrightarrow \underline{\omega_j = \omega^{k_j}}$ for some $k_j \in \{0, 1, \dots, n-1\}$

ω_j uniquely determines ρ_j, since $\rho_j(t^s) = (\rho_j(t))^s = \omega_j^s$.

So $\omega_i \neq \omega_j$ for $i \neq j$. Thus all the k_j should be different. So they should take all the values from $\{0, 1, \dots, n-1\}$.

Conclusion: one can reorder the ρ_j to get $\omega_j = \omega^j$, that is,

$\rho_j(t^s) = \omega^{js}$.

Summarising, one gets

The character table for C_n:

	$\chi \backslash \mathcal{C}$	$[1]$	$[t]$	$[t^2]$	$\dots$	$[t^{n-1}]$
	$\#\mathcal{C}$	1	1	1	$\dots$	1
irreps	$V^{tr} = V_0$	1	1	1		1
	V_1	1	ω	ω^2		ω^{n-1}
	V_2	1	ω^2	$(\omega^2)^2$	$\dots$	$(\omega^2)^{n-1}$
	$\vdots$					
	V_{n-1}	1	ω^{n-1}	$(\omega^{n-1})^2$		$(\omega^{n-1})^{n-1}$
$\bigoplus_{i=0}^{n-1} V_i =$	V^{reg}	n	0	0	$\dots$	0

Let us check that indeed $\sum_{i=0}^{n-1} \chi^{V_i} = \chi^{V^{reg}}$

For $1 \le s \le n-1$, $1 + \omega^s + (\omega^2)^s + \dots + (\omega^{n-1})^s = 1 + \omega^s + (\omega^s)^2 + \dots + (\omega^s)^{n-1} =$

$= \dfrac{1-(\omega^s)^n}{1-\omega^s} = \dfrac{1-(\overbrace{\omega^n}^{=1})^s}{1-\underbrace{\omega^s}_{\neq 1}} = 0$, as desired.

We will next see that the property of having irreps of degree 1 only (like for C_n) characterises abelian groups:

Thm 8: $\boxed{\text{A finite group is abelian iff all its irreps are of degree 1}}$

□ A finite G is abelian $\Leftrightarrow$ all its conjugacy classes are 1-element

$\Leftrightarrow \#\mathrm{Conj}(G) = \#G \Leftrightarrow \#\mathrm{Irrep}(G) = \#G \Leftrightarrow \forall V_i \in \mathrm{Irrep}(G),\ \dim_{\mathbb{C}}(V_i) = 1$

$$\#\mathrm{Irrep}(G) \le \sum_{i=1}^{\#\mathrm{Irrep}(G)} \underbrace{(\dim_{\mathbb{C}}(V_i))^2}_{\ge 1} = \#G,$$

with equality $\Leftrightarrow$ all $\dim_{\mathbb{C}}(V_i) = 1$. ⊠

<u>Prop. 9</u>: Let a finite group G have an abelian subgroup H.

Then $\forall V_i \in \mathrm{Irrep}(G)$, $\dim_{\mathbb{C}}(V_i) \leq \underbrace{\#G/\#H}_{\text{index of } H \text{ in } G}$.

$\square$ $(V,\rho) \in \mathrm{Irrep}(G) \Rightarrow (V, \rho\iota) \in \mathrm{Rep}(H)$, where $\iota: H \to G$ is the group inclusion.

H abelian $\Rightarrow V \cong \bigoplus m_i V_i$, where $V_i \in \mathrm{Irrep}(H)$ all have degree 1.

So $\exists v \in V$ & $\alpha_h \in \mathbb{C}$ (for all $h \in H$) s.t. $\forall h \in H$, $h \cdot v = \alpha_h v$.

$r := \#G/\#H$, $G = g_0 H \sqcup g_1 H \ldots \sqcup g_{r-1} H$ for some $g_i \in G$ (decomposition into cosets).

$\forall g \in G$ writes as $g_i h$ for some i & some $h \in H \Rightarrow$

$g \cdot v = g_i \cdot (h \cdot v) = g_i \cdot (\alpha_h v) = \alpha_h\, g_i \cdot v$.

Thus r vectors $g_0 \cdot v, \ldots, g_{r-1} \cdot v$ span the same vector space as all $g \cdot v$, $g \in G$. This is a sub-rep. V' of V, since $g' \cdot (g \cdot v) = (g'g) \cdot v$.

It is non-zero ($v \in V'$), and V is an irrep. So $V' = V$.

Conclusion: $r \geq \dim_{\mathbb{C}} V' = \dim_{\mathbb{C}} V$. $\boxtimes$

<u>Ex.</u>: S_3 has a sub-group $\underbrace{\{Id, (123), (132)\}}_{\simeq C_3}$ of index $\frac{\#S_3}{3} = \frac{6}{3} = 2$.

So all its irreps are of degree ≤ 2. As we have seen last time, this is indeed the case.