6. Character Tables: an Example & an Algorithm

We have seen that if you know the characters of all the irreps of a finite group, then you have a rather complete and very efficient description of all its reps.
Today we'll compute all the χ^{V_i}, $V_i \in \mathrm{Irrep}(S_3)$. This data is organised into convenient character tables.

The first step in the construction of character tables is an enumeration of conjugacy classes. We'll now show how to do it for all symmetric groups S_n.

Thm 7: $\boxed{\text{In } S_n,\ \sigma \sim \sigma' \Leftrightarrow \text{ they have the same cycle decomposition, i.e., } \forall k, \text{ the same number of } k\text{-cycles.}}$

$\square$ This follows from the following statement:
for σ with a cycle decomposition $\sigma = (i_{1,1} \ldots i_{1,k_1})(i_{2,1} \ldots i_{2,k_2}) \ldots$
and for any $\tau \in S_n$, $\tau\sigma\tau^{-1} = (\tau(i_{1,1}) \ldots \tau(i_{1,k_1}))(\tau(i_{2,1}) \ldots \tau(i_{2,k_2})) \ldots$ $(*)$.
To see this, it suffices to check that both sides send a $j \in \{1,2,\ldots,n\}$ to the same element of $\{1,2,\ldots,n\}$. Write j as $\tau(i_{s,t})$ for some s,t.
Then $\tau(i_{s,t}) \xmapsto{\tau^{-1}} i_{s,t} \xmapsto{\sigma} i_{s,t+1} \xmapsto{\tau} \tau(i_{s,t+1})$ (here k_s+1 means 1).
But this is precisely the image of $j = \tau(i_{s,t})$ by the RHS of $(*)$ $\blacksquare$

We are now ready to describe $\mathrm{Conj}(S_3)$:

- $\mathcal{C}_1 = \{\mathrm{Id}\}$
- $\mathcal{C}_2 = \{(12), (23), (13)\}$
- $\mathcal{C}_3 = \{(123), (132)\}$

$\Sigma \#\mathcal{C}_i = 6 = 3! = \#S_3$

$\#\mathcal{C}$	1	3	2
$\chi \backslash \mathcal{C}$	$[\mathrm{Id}]$	$[(12)]$	$[(123)]$

So $\#\mathrm{Irrep}(S_3) = \#\mathrm{Conj}(S_3) = 3$.
$d_i := \dim_{\mathbb{C}}(V_i)$.
$\sum_{i=1}^{3} d_i^2 = \#S_3 = 6 \Rightarrow d_1 = d_2 = 1,\ d_3 = 2$

As V_1 it is common to choose V^{tr}, with $\chi^{V^{tr}}(g) = 1$ for all g.

$\chi \backslash \mathcal{C}$	$[\mathrm{Id}]$	$[(12)]$	$[(123)]$
V^{tr}	1	1	1

What is the 2nd irrep of S_3 of degree 1?

$(V_2 \cong \mathbb{C}, \rho_2)$, $\rho_2 : S_3 \xrightarrow[grp]{} \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^*$. Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

$\chi^{V_2} = tr \circ \rho_2 = \rho_2 \Rightarrow \rho_2 \in CF(S_3)$.

Put $\rho_2((12)) = \varepsilon$. Then $\Downarrow$ $\rho_2((23)) = \rho_2((13)) = \varepsilon$.

$(12)^2 = Id \Rightarrow 1 = \rho_2(Id) = \rho_2((12)^2) = (\rho_2(12))^2 = \varepsilon^2 \Rightarrow \varepsilon = \pm 1$

$(123) = (12)(23) \Rightarrow \rho_2((123)) = \rho_2((12))\,\rho_2((23)) = \varepsilon^2 = 1$

(We used that ρ_2 is a group morphism.)

$\rho_2((132)) = \rho_2((123)) = 1$.

The choice $\varepsilon = 1$ yields V^{tr}, and $\varepsilon = -1$ yields the desired (V_2, ρ_2).

This irrep has an analogue for all S_n, called the **sign representation**:

$V^{sgn} \cong \mathbb{C}$, $\sigma \cdot v = sgn(\sigma) v$,

where $sgn(\sigma) = \begin{cases} 1, & \sigma \text{ is even}, \\ -1, & \sigma \text{ is odd}. \end{cases}$

χ \ $\mathcal{C}$	$[Id]$	$[(12)]$	$[(123)]$
V^{sgn}	1	-1	1

We are left with (V_3, ρ_3) of degree 2.

3 methods of computing χ^{V_3} will be given; for more complicated groups, they can all be useful.

(1) $V^{reg} \cong V^{tr} \oplus V^{sgn} \oplus 2V_3 \Rightarrow \chi^{V_3} = \frac{1}{2}(\chi^{V^{reg}} - \chi^{V^{tr}} - \chi^{V^{sgn}})$

$\chi^{V^{reg}}(\sigma) = \# S_3\, \delta_{\sigma,1} = 6\,\delta_{\sigma,1}$

(2) $V^{perm} \cong V^{tr} \oplus V^{st}$ (seen in Lecture 2)

$V^{perm} = \bigoplus_{i=1}^{3} \mathbb{C} e_i$, $\sigma \cdot e_i = e_{\sigma(i)}$,

$V^{st} = \mathrm{Ker}\, \varepsilon \subset V^{perm}$, $\varepsilon : V^{perm} \to \mathbb{C}$, $e_i \mapsto 0$. $\dim_{\mathbb{C}} V^{st} = 2$, a basis: $e_1 - e_3 =: e_{13}$, $e_2 - e_3 =: e_{23}$

We've seen that V^{st} is irreducible (Lecture 2).

But we've just shown that S_3 has only one irrep. of degree 2 (up to isomorphism),

so $V_3 \cong V^{st}$.

This gives an explicit description of V_3, which can be used to compute its character:

$\to (12)\cdot e_{13} = (12)(e_1 - e_3) = e_2 - e_3 = e_{23} = 0\cdot e_{13} + 1\cdot e_{23}$

$(12)\cdot e_{23} = (12)(e_2 - e_3) = e_1 - e_3 = e_{13} = 1\cdot e_{13} + 0\cdot e_{23}$

$$\Rightarrow \chi^{V^{st}}((12)) = tr\, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{the matrix of } \rho((12)) \text{ in the basis } (e_{13},\, e_{23}).} = 0$$

$\to (123)\cdot e_{13} = e_2 - e_1 = (e_2 - e_3) - (e_1 - e_3) = e_{23} - e_{13}$

$(123)\cdot e_{23} = e_3 - e_1 = -e_{13}$

$$\Rightarrow \chi^{V^{st}}((123)) = tr \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = -1.$$

(3) In (2) one could do without explicit matrices:

$\chi^{V^{st}} = \chi^{V^{perm}} - \chi^{V^{tr}}$

$\underbrace{\chi^{V^{perm}}}_{\chi^{perm}}(\sigma) = \#\{1,2,3\}^{\sigma}$ (elements of $\{1,2,3\}$ fixed by σ)

↑ seen in Lecture 4.

$\chi^{perm}(Id) = 3$,

$\chi^{perm}((12)) = 1$,

$\chi^{perm}((123)) = 0$.

Summarising, one gets

The character table for S_3:

		$[Id]$	$[(12)]$	$[(123)]$	
	$\#\mathcal{C}$	1	3	2	} $6 = \#S_3$
	$\chi \setminus \mathcal{C}$	$[Id]$	$[(12)]$	$[(123)]$	
$Irrep(S_3)$	V^{tr}	1	1	1	compulsory part
	V^{sgn}	1	-1	1	
	V^{st}	2	0	-1	
$V^{tr}\oplus V^{sgn}\oplus 2V^{st} =$	V^{reg}	6	0	0	auxiliary part
$V^{tr}\oplus V^{st} =$	V^{perm}	3	1	0	
		$\dim_{\mathbb{C}} V$			

Double-cheking:

- $(\chi^{st}, \chi^{st}) = \frac{1}{6}\sum_{\sigma\in S_3} \chi^{st}(\sigma)\overline{\chi^{st}(\sigma)} = \frac{1}{6}\sum_{\mathcal{C}\in Conj(S_3)} \#\mathcal{C}\cdot |\chi^{st}(\underbrace{\sigma_{\mathcal{C}}}_{\text{any element of } \mathcal{C}})|^2 = \frac{1}{6}(1\cdot 2^2 + 3\cdot 0^2 + 2\cdot(-1)^2) = 1.$
- $(\chi^{st}, \chi^{sgn}) = \frac{1}{6}\sum_{\mathcal{C}\in Conj(S_3)} \#\mathcal{C}\cdot \chi^{st}(\sigma_{\mathcal{C}})\cdot\overline{\chi^{sgn}(\sigma_{\mathcal{C}})} = \frac{1}{6}(1\cdot 2\cdot \overline{1} + 3\cdot 0\cdot\overline{(-1)} + 2\cdot(-1)\cdot 1) = 0.$

We get the values predicted by the orthonormality (Prop. 7).

Algorithm for computing character tables of G:

(1) Enumerate $\mathrm{Conj}(G) = \{\mathcal{C}_1, \dots, \mathcal{C}_{c(G)}\}$. Compute $\#\mathcal{C}_i$ for all i.

(2) Determine all irreps $V_1, \dots, V_s$ of degree 1, including $V_1 = V^{tr}$.

(3) Recall other irreps $V_{s+1}, \dots, V_{s+t}$ you might know (e.g., by looking at permutation reps).

(4) Add irreps $V_i \otimes V_{s+j}$, $1 \le i \le s$, $1 \le j \le t$, if they are not isomorphic to the ones already in your list (i.e., if their characters are different). This gives $V_{s+t+1}, \dots, V_{s+t+u}$.

To be seen in your homework and later lectures

(5) Determine $\dim_{\mathbb{C}}(V_i)$ for the remaining i (i.e., $i > s+t+u$), using

- $\sum_{i=1}^{c(G)} (\dim_{\mathbb{C}}(V_i))^2 = \#G$
- $\dim_{\mathbb{C}}(V_i)$ divides $\#G$

(6) Fill in the rows corresponding to $V_1, \dots, V_{s+t+u}$, and the column for $\mathcal{C}_1 = [1]$, using $\chi^{V_i}(1) = \dim_{\mathbb{C}}(V_i)$.

(7) Fill in the remaining cells, using

- $\sum_{i=1}^{c(G)} \dim_{\mathbb{C}}(V_i)\, \chi^{V_i}(g) = 0$ for all $g \neq 1$.
- $\sum_{\mathcal{C} \in \mathrm{Conj}(G)} \#\mathcal{C}\, \chi_i(\mathcal{C}) \overline{\chi_j(\mathcal{C})} = \#G \cdot \delta_{ij}$

  Here $\chi(\mathcal{C}) := \chi(g)$ for any $g \in \mathcal{C}$.
- $\chi_i(g^{-1}) = \overline{\chi_i(g)}$
- $\sum_{i=1}^{c(G)} \chi^{V_i}(g) \overline{\chi^{V_i}(g')} = \mathbb{1}_{g \sim g'} \cdot \frac{\#G}{\#[g]}$ ← To be seen in a later lecture.