Lectures 6 and 7: Inverse functions

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MA1S11A: Calculus with Applications for Scientists

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1 Bricks and mortar

Most function you'll encounter in this module are obtained from the "elementary bricks":

- ✓ c, $c \in \mathbb{R}$ (constant functions);
- ✓ x;
- ✓ $\sin(x)$;
- ✓ e^x (exponential function)

using as "mortar" different operations:

- ✓ arithmetic operations: $f + g$, $f - g$, fg, $\frac{f}{g}$;
- ✓ powers: f^{α}, $\alpha \in \mathbb{R}$;
- ✓ composition: $f \circ g$;
- ✓ inverse function f^{-1};
- ✓ gluing functions from different pieces (piecewise defined functions).

Today we'll study the inverse functions.

2 "Undoing" operations

To undo the effect of addition, we subtract: if we replace f by $f+g$, then to get back f, we subtract g:

$$(f+g)-g=f.$$

To undo the effect of multiplication, we divide: if we replace f by fg, then to get back f, we divide by g:

$$(fg)/g=f.$$

In this case, it is not always possible to undo the effect: if $g(x)=0$ for some x, then we cannot recover f by division.

We shall now discuss how to undo the effect of composition (when possible).

3 The Celsius–Fahrenheit example

Let us consider an example of two different temperature scales, those of Celsius (1742) and of Fahrenheit (1724).

	water freezing	water boiling
Celsius	0°	100°
Fahrenheit	32°	212°

(0°F refers to the freezing point of brine, the lowest temperature Fahrenheit could reliably reproduce.)

If f is the value of temperature on the Fahrenheit scale, and c is its value on the Celsius scale, then

$$f = \frac{9}{5}c + 32.$$

3 The Celsius–Fahrenheit example

Suppose that we know the temperature on the Fahrenheit scale, and want to convert it to Celsius. (Someone from the US told us the temperature at their home, and we want to figure out what they mean saying that it's 80°!)

We view the conversion formula above as an equation for c, and solve it:

$$f = \frac{9}{5}c + 32,$$

$$f - 32 = \frac{9}{5}c,$$

$$c = \frac{5}{9}(f - 32).$$

In this case, we knew f as a function of c, and we have been able to reverse the procedure, and compute c as a function of f.

If we had a different functional dependence, for example,

$$f = \frac{9}{5}c^2 + 32,$$

then reconstruction of c as a function of f would be impossible: $c^2 = \frac{5}{9}(f - 32)$ yields two possible values for c.

4 Definition

Definition. Suppose that for a function f there exists a function g such that

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g,$$
$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

Then g is said to be the **inverse** of f, denoted by $g = f^{-1}$. Also, f and g are called **inverse functions**.

This definition is symmetric with respect to f and g, so $(f^{-1})^{-1} = f$ (the inverse of the inverse of f is f itself).

⚠ Whenever f is the name of a function, f^{-1} will denote the inverse, and -1 will never mean the exponent: $f^{-1}(x) \neq \frac{1}{f(x)} = f(x)^{-1}$. This is a rather common mistake, try to refrain from it!

Similarly, $f^2(x) = f(f(x))$, while $f(x)^2 = f(x)f(x)$.

5 Examples

Example 1. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$.
Then $f(g(x)) = (\sqrt{x})^2 = x$ for all x in the domain of g (that is, $x \geqslant 0$).
However, $g(f(x)) = \sqrt{x^2} = |x|$, which is different from x for $x < 0$ (which still is in the domain of f).
⚠ Therefore f and g are not inverse functions!

Example 1 bis. Let $f(x) = x^2$ defined for $x \geqslant 0$, and $g(x) = \sqrt{x}$.
Then $f(g(x)) = (\sqrt{x})^2 = x$ for all x in the domain of g (that is, $x \geqslant 0$).
Also, $g(f(x)) = \sqrt{x^2} = x$ for $x \geqslant 0$ (which is the domain of f).
Therefore f and g are inverse functions.

So, restricting the domain may change the property of being inverse.

5 Examples

Example 2. The function $f(x) = x^3$ has an inverse $f^{-1}(x) = \sqrt[3]{x}$. Indeed,

$$f(f^{-1}(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x,$$

$$f^{-1}(f(x)) = f^{-1}\left(x^3\right) = \sqrt[3]{x^3} = x$$

for all real x.

That is why the roots were absent from the list of "bricks"!

Example 3. For $c \neq 0$, the function $f(x) = cx$ has an inverse $f^{-1}(x) = \frac{1}{c}x$:

$$f(f^{-1}(x)) = f\left(\frac{1}{c}x\right) = c \cdot \frac{1}{c}x = x,$$

$$f^{-1}(f(x)) = f^{-1}(cx) = \frac{1}{c} \cdot cx = x$$

for all real x.

5 Examples

Example 4. In Lecture 2, we wanted to compute the range of $f(x) = \frac{x+1}{x-1}$.
For that, we solved the equation $\frac{x+1}{x-1} = y$ for x. The answer was $x = \frac{y+1}{y-1}$ for $y \neq 1$.
In other words, the way y depends on x is the same as the way x depends on y. In particular, this function f is the inverse of itself:

$$f(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-(x-1)}{x-1}} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = \frac{2x}{2} = x$$

for all $x \neq 1$.

6 The inverse is unique

We will now establish key properties of inverse functions.

Theorem. If a function f has an inverse, then that inverse is unique.

In particular, we have the right to talk about THE inverse of f, and the notation f^{-1} is not ambiguous.

Remark. In maths, we use different words to name our results: *Theorem, Proposition, Lemma, Corollary*. The name choice depends on the importance of the statement, and on its connexion with other results. In this course for simplicity we'll mainly use the word *Theorem*.

6 The inverse is unique

Theorem. If a function f has an inverse, then that inverse is unique.

Proof (by contradiction).
Suppose that g_1 and g_2 are two different inverses of f. Then

$$g_1(f(g_2(x))) = g_1(x) \text{ because } f(g_2(x)) = x$$

$$g_1(f(g_2(x))) = g_2(x) \text{ because } g_1(f(x)) = x.$$

Both equalities hold for all x in the domain of g_2. In particular, this means that $g_1(x)$ is defined for all x for which $g_2(x)$ is defined.

Analysing $g_2(f(g_1(x)))$ in a similar way, one gets a symmetric statement: $g_2(x)$ is defined for all x for which $g_1(x)$ is defined.

This means that the domains of g_1 and g_2 coincide, and that $g_1(x) = g_2(x)$ for all x from this common domain.

Thus the functions g_1 and g_2 coincide, contradiction. □

7 Domain and range of the inverse function

Theorem. Suppose that a function f has an inverse. Then the domain of f is equal to the range of f^{-1}, and the range of f is equal to the domain of f^{-1}.

Proof. To show that the domain of f is equal to the range of f^{-1}, it is sufficient to show that every x from the domain of f is in the range of f^{-1}, and every x from the range of f^{-1} is in the domain of f.

(More generally, to show that some sets A and B coincide, it suffices to check that $x \in A$ implies $x \in B$, and vice versa.)

First, suppose that x is in the domain of f. Then $f^{-1}(f(x)) = x$, so x is the value of f^{-1} at the point $f(x)$, and therefore x is in the range of f^{-1}.

Now, suppose that x is in the range of f^{-1}, so that $x = f^{-1}(t)$ for some t in the domain of f^{-1}. Then $t = f(f^{-1}(t)) = f(x)$, so f is defined at the point x, and therefore x is in the domain of f.

The proof of the other half of the theorem (the range of f is equal to the domain of f^{-1}) is exactly the same with f and f^{-1} interchanged. $\square$

8 A symmetry between f and f^{-1}

Theorem. Suppose that a function f has an inverse. Then $y = f(x)$ for some real x and y iff $x = f^{-1}(y)$:

$$y = f(x) \qquad \Longleftrightarrow \qquad x = f^{-1}(y).$$

Exercise. Prove the theorem. Take inspiration from the proof from the previous slide.

From this result we deduce a algorithm for computing the inverse function (when it exists):

1. Form the equation $y = f(x)$.
2. If possible, solve that equation for x as a function of y, $x = g(y)$.
3. If x is required as the name of the independent variable, interchange the names x and y: $y = g(x)$.
4. Describe the domain of g (which is the range of f).

8 A symmetry between f and f^{-1}

Example. Let $f(x) = \sqrt{3x-2}$. Its natural domain is $[\frac{2}{3}, +\infty)$.

1. We form the equation $y = \sqrt{3x-2}$.
2. Solving this equation for x goes as follows:

$$y = \sqrt{3x-2},$$
$$y^2 = 3x-2,$$
$$x = \frac{y^2+2}{3}.$$

3. Interchanging names of variables, we obtain the formula $g(x) = \frac{x^2+2}{3}$.
4. The domain of f^{-1} is equal to the range of f, which is $[0, +\infty)$.

Conclusion: We get the inverse

$$f^{-1}(x) = \frac{x^2+2}{3}, \quad x \geqslant 0.$$

8 A symmetry between f and f^{-1}

Theorem. Suppose that a function f has an inverse. Then $y = f(x)$ for some real x and y iff $x = f^{-1}(y)$:

$$y = f(x) \qquad \Longleftrightarrow \qquad x = f^{-1}(y).$$

From this result we also deduce

Theorem. Suppose that a function f has an inverse. Then the graph of f^{-1} is obtained from that of f by reflection about the line $y = x$.

8 A symmetry between f and f^{-1}

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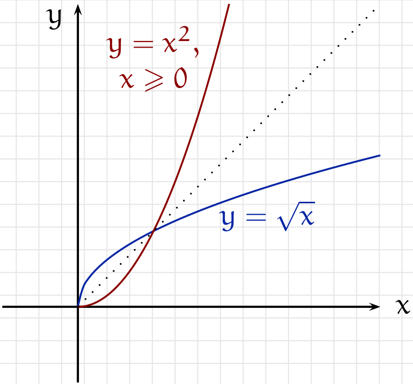
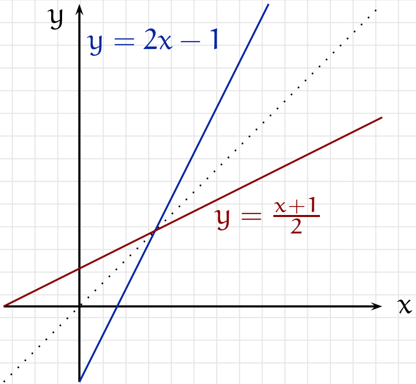
Examples.





9 When does a function have an inverse?

Definition. A function f is called **injective**, or **one-to-one**, if its values at any distinct points from its domain are distinct:

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Theorem. A function admits an inverse iff it is injective.

Proof. 1) Suppose that f has an inverse f^{-1}, and that $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$ from its domain. Then

$$x_1 = f^{-1}(f(x_1)) = f^{-1}(f(x_2)) = x_2,$$

which is a contradiction.

2) Next, suppose that f is injective. This means that for any y from its range, there is exactly one pre-image x, i.e., $y = f(x)$. Put $g(y) = x$. This defines a function on the range of f.

✓ By construction, $g(f(x)) = x$ for all x from the domain of f.

✓ Also, any x from the domain of g is in the range of f, and so has the form $x = f(t)$ for some t. So,

$$f(g(x)) = f(g(f(t))) = f(t) = x.$$

Therefore, g is the inverse of f. □

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Theorem. A function admits an inverse iff it is injective.

This yields the **horizontal line test** for invertibility: a function is invertible iff each horizontal line meets its graph at most once. This is just a reflection of the vertical line test about the line $y = x$!

Example 1. The absolute value $f(x) = |x|$ is not injective, since $f(-1) = 1 = f(1)$. Therefore, it does not have an inverse.

More generally, an even function never has an inverse, since $f(-x) = f(x)$.

Example 2. The function $f(x) = 2\sin(\frac{x}{3} - 1)$ is not injective, since $f(0) = f(6\pi)$. Therefore, it does not have an inverse.

More generally, a periodic function never has an inverse, since $f(x) = f(x + T)$.

9 When does a function have an inverse?

In both examples above we used the "negative" part of the theorem. To use the "positive" part, one more definition is helpful.

Definition. A function f is called

- ✓ **increasing** if for any x_1, x_2 from its domain,
$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2);$$
- ✓ **decreasing** if for any x_1, x_2 from its domain,
$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2);$$
- ✓ **monotonous** if it is increasing or decreasing.

Theorem. A monotonous function is injective and admits an inverse.

Exercise. Prove the theorem.

Example 1. The linear function $f(x) = mx + b$ with $m \neq 0$ is monotonous: $x_1 < x_2$ implies $mx_1 + b < mx_2 + b$ when $m > 0$, and $mx_1 + b > mx_2 + b$ when $m < 0$. Hence, all these functions admit inverses.

When does a function have an inverse?

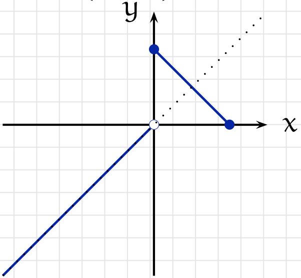
Example 2. The function

$$f(x) = \begin{cases} x, & x < 0 \\ 1 - x, & 0 \leqslant x \leqslant 1 \end{cases}$$

is not monotonous, since $f(-1) = -1 < f(0) = 1 > f(1) = 0$. However, it is invertible, with $f^{-1} = f$:

$$f(f(x)) = f(x) = x \qquad \text{for } x < 0,$$

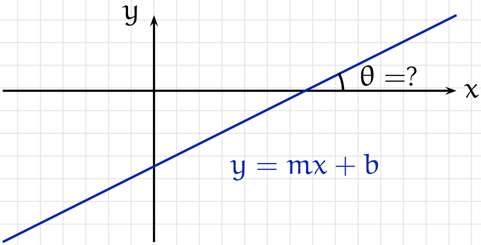
$$f(f(x)) = f(1 - x) = 1 - (1 - x) = x \qquad \text{for } 0 \leqslant x \leqslant 1.$$



⚠ So, the converse of the theorem is false: invertible $\nRightarrow$ monotonous.

10 Inverse trigonometric functions

All trigonometric functions are periodic, hence not invertible. However, it would be nice to have some kind of $\sin^{-1}$, $\tan^{-1}$ etc., for instance in order to determine the angles between two lines defined by equations:

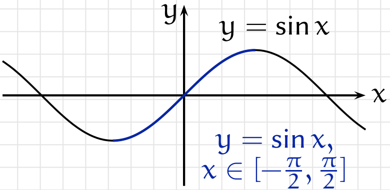


We would like to write $\theta = \tan^{-1}(m)$, and use a calculator to compute it!

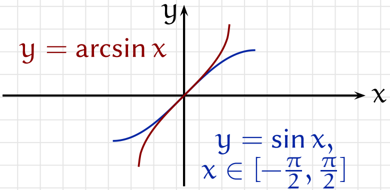
A way out is to restrict the domain of trigonometric functions.

10 Inverse trigonometric functions

The function sin is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$:



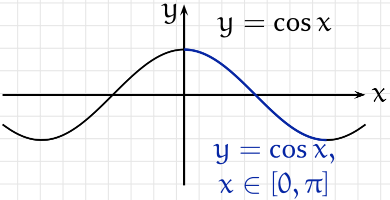
Its range is $[-1, 1]$. Hence, its restriction to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ has an inverse, denoted by $\arcsin = \sin^{-1}$, and called **arcsine**.



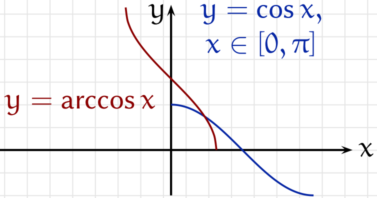
Its domain is $[-1, 1]$, and its range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. It is odd, since sin is odd, but NOT periodic!

10 Inverse trigonometric functions

Similarly, cos is decreasing on $[0, \pi]$:



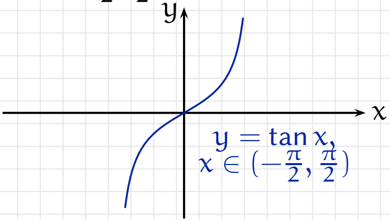
Its range is $[-1, 1]$. Hence, its restriction to $[0, \pi]$ has an inverse, denoted by $\arccos = \cos^{-1}$, and called **arccosine**.



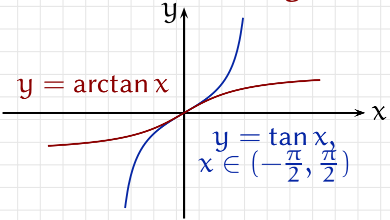
Its domain is $[-1, 1]$, and its range is $[0, \pi]$.

10 Inverse trigonometric functions

Finally, tan is increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$:



Its range is $\mathbb{R} = (-\infty, \infty)$. Hence, its restriction to $(-\frac{\pi}{2}, \frac{\pi}{2})$ has an inverse, denoted by $\arctan = \tan^{-1}$, and called **arctangent**.



Its domain is $\mathbb{R}$, and its range is $(-\frac{\pi}{2}, \frac{\pi}{2})$. It is odd, since tan is odd.

10 Inverse trigonometric functions

You can think about the inverse trigonometric functions as follows:

✓ $\arcsin(x)$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sin is x;

✓ $\arccos(x)$ is the angle between 0 and π whose cos is x;

✓ $\arctan(x)$ is the angle strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tan is x.

⚠ If you write "$\arcsin(x)$ is the angle whose sin is x", this is a mistake: there are plenty of angles with the same sin!

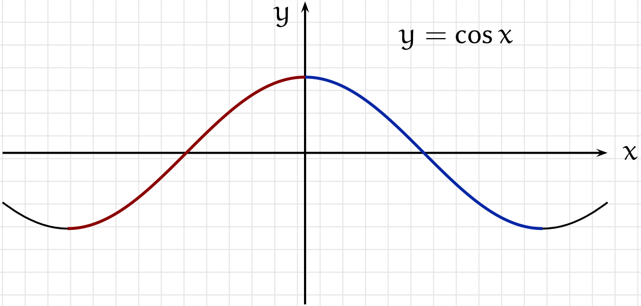
function	domain	range	symmetries
arcsin	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	odd
arccos	$[-1, 1]$	$[0, \pi]$	–
arctan	$\mathbb{R}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	odd
arccot	$\mathbb{R}$	$(0, \pi)$	–

⚠ You know that when you want to determine the natural domain of a function, you need to be careful with expressions $\sqrt{g(x)}$ and $\frac{h(x)}{g(x)}$. Now, one more type of expressions should be taken care of: when you see $\arcsin(g(x))$ or $\arccos(g(x))$, always check that $-1 \leqslant g(x) \leqslant 1$.

10 Inverse trigonometric functions

function	domain	range	symmetries
arcsin	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$	odd
arccos	$[-1,1]$	$[0,\pi]$	–
arctan	$\mathbb{R}$	$(-\frac{\pi}{2},\frac{\pi}{2})$	odd
arccot	$\mathbb{R}$	$(0,\pi)$	–

If, for instance, you are looking for an angle between $-\pi$ and 0 whose cos is x, the answer will be $-\arccos(x)$.



11 Digression: writing maths

In this module I am being rather informal in writing down mathematical arguments. For example, when solving the equation $\frac{x+1}{x-1} = a$ for x, I wrote

$$\frac{x+1}{x-1} = a,$$

$$x + 1 = ax - a,$$

$$a + 1 = ax - x = x(a - 1),$$

$$x = \frac{a+1}{a-1}.$$

Here, some lines are equivalent to the preceding ones, and some are simply a consequence. Thus,

- ✓ adding the same element to both sides of the equation leads to an equivalent equation;
- ✓ multiplying both sides by the same thing leans to a consequence only, since you might multiply by zero ($a = b$ implies $a * 0 = b * 0$, but not the other way round).

So, our reasoning can be written more precisely:

$$\begin{aligned} & \frac{x+1}{x-1} = a, \\ \implies \quad & x + 1 = ax - a, \\ \iff \quad & a + 1 = ax - x = x(a-1), \\ \impliedby \quad & x = \frac{a+1}{a-1}. \end{aligned}$$

Then, to replace the implication signs by equivalences, you should mention that $x = 1$ is impossible when $x + 1 = ax - a$, and $a = 1$ is impossible when $a + 1 = x(a-1)$.

⚠ I recommend these notations. They'll make life easier for your graders and for yourself (you'll have less chances to forget to think about the division be zero etc.).

11 Digression: writing maths

Another unfortunate notation I often see in students' work is the comma replacing either *and* or *or*:

$$x(x-2)=0 \iff x=0, x=2;$$

$$x^2+(y-1)^2=0 \iff x=0, y=1.$$

Here the comma means *or* in the first case, *and* in the second case.

Guess what percentage of students forget what their comma meant three lines later?

⚠ Write the words *and* / *or* when you mean them! They are not that long!

12 Digression: infinity

One more common mistake I'd like to point out concerns the notations $+\infty$ and $-\infty$. These are not real numbers, so you cannot include them in the closed intervals:

⚠ Never write $[0, +\infty]$ etc.!!! Otherwise you allow real numbers x satisfying $x = +\infty$, but $+\infty$ is not a real number!

Recall that $[0, +\infty)$ means the ray of all real numbers x satisfying $x \geqslant 0$:

$$[0, +\infty) = \{x \in \mathbb{R} \mid 0 \leqslant x\}.$$