5. Tracing Representations: Characters. II

Among the properties of characters we've seen, the key one is $\chi(hgh^{-1}) = \chi(g)$. It suggests the importance of the conjugation relation in representation theory.

Def.: The conjugation relation in $\mathcal{G}$ is defined by

$$g \sim g' \Leftrightarrow g' = hgh^{-1} \text{ for some } h.$$

It is an equivalence relation. Indeed, one easily checks:

1) reflexivity: $g \sim g$;

2) symmetry: $g \sim g' \Leftrightarrow g' \sim g$;

3) transitivity: $g \sim g',\ g' \sim g'' \Rightarrow g \sim g''$

Hence a decomposition $\mathcal{G} = \mathcal{C}_1 \sqcup \ldots \sqcup \mathcal{C}_{c(\mathcal{G})}$ into equivalence classes, called conjugacy classes: $g \sim g' \Leftrightarrow g$ & g' belong to the same $\mathcal{C}_i$.

$\mathrm{Conj}(\mathcal{G}) := \{\mathcal{C}_1, \ldots, \mathcal{C}_{c(\mathcal{G})}\}$.

Def.: $CF(\mathcal{G}) := \{\varphi : \mathcal{G} \to \mathbb{C} \mid \underline{\forall g, h \in \mathcal{G},\ \varphi(hgh^{-1}) = \varphi(g)}\ (*)\}$.

Such φ are called class (= central) functions.

Exo: $(*) \Leftrightarrow \varphi$ is constant on all $\mathcal{C} \in \mathrm{Conj}(\mathcal{G})$ $\Rightarrow$ term "class f."

$\Leftrightarrow \forall a, b \in \mathcal{G},\ \varphi(ab) = \varphi(ba)$. $\Rightarrow$ term "central f."

Ex.: $\boxed{1}$ $\forall V \in \mathrm{Rep}(\mathcal{G})$, $\underline{\chi^V \in CF(\mathcal{G})}$.

$\boxed{2}$ $\forall \mathcal{C} \in \mathrm{Conj}(\mathcal{G})$, $\mathbb{1}_\mathcal{C}(g) := \begin{cases} 1, & g \in \mathcal{C} \\ 0, & g \notin \mathcal{C} \end{cases}$, $\underline{\mathbb{1}_\mathcal{C} \in CF(\mathcal{G})}$.

We'll use the easy example $\boxed{2}$ to get insight into $\boxed{1}$, i.e., into reps and chars.

From now on, G is finite.

Prop. 6: (1) $\mathrm{Maps}(G,\mathbb{C})$ is a vector space over $\mathbb{C}$, the operations being defined by

$$(\varphi+\psi)(g)=\varphi(g)+\psi(g), \qquad \forall \varphi,\psi: G\to\mathbb{C}$$
$$(\alpha\varphi)(g)=\alpha\varphi(g). \qquad \forall \alpha\in\mathbb{C}$$

(2) $\boxed{(\varphi,\psi):=\frac{1}{\#G}\sum_{g\in G}\varphi(g)\overline{\psi(g)}}$ is an inner product on $\mathrm{Maps}(G,\mathbb{C})$.

(3) $CF(G)$ is a lin. sub-space of $\mathrm{Maps}(G,\mathbb{C})$, for the structure from (1).

(4) $\mathbb{1}_{\mathcal{C}}$, $\mathcal{C}\in\mathrm{Conj}(G)$, form a basis of $CF(G)$.

□ (1) & (3) are obvious.

(2) All the inner product axioms are easy to check:

- $(\psi,\varphi)=\overline{(\varphi,\psi)}$
- $(\varphi+\varphi',\psi)=(\varphi,\psi)+(\varphi',\psi)$
- $(\alpha\varphi,\psi)=\alpha(\varphi,\psi)$
- $(\varphi,\varphi)=\frac{1}{\#G}\sum_{g\in G}\varphi(g)\overline{\varphi(g)}=\frac{1}{\#G}\sum_{g\in G}|\varphi(g)|^2\geq 0,$

$(\varphi,\varphi)=0 \Leftrightarrow \forall g\in G,\ \varphi(g)=0 \Leftrightarrow \varphi=0.$

(4) • Lin. independence:

$\sum_{\mathcal{C}\in\mathrm{Conj}(G)}\alpha_{\mathcal{C}}\mathbb{1}_{\mathcal{C}}$ for some $\alpha_{\mathcal{C}}\in\mathbb{C} \Rightarrow \forall g\in G,\ 0=\sum\alpha_{\mathcal{C}}\mathbb{1}_{\mathcal{C}}(g)=\alpha_{[g]}$

the conj. class of g

$\Rightarrow \forall\mathcal{C},\ \alpha_{\mathcal{C}}=0.$

• Spanning property:

Choose a representative $g_{\mathcal{C}}$ of any $\mathcal{C}\in\mathrm{Conj}(G)$.

Let us check that $\forall\varphi\in CF(G)$ writes as $\varphi=\sum_{\mathcal{C}\in\mathrm{Conj}(G)}\underbrace{\varphi(g_{\mathcal{C}})}_{\in\mathbb{C}}\mathbb{1}_{\mathcal{C}}$.

It suffices to evaluate both sides on each $h\in G$:

$(\sum\varphi(g_{\mathcal{C}})\mathbb{1}_{\mathcal{C}})(h)=\sum\varphi(g_{\mathcal{C}})\mathbb{1}_{\mathcal{C}}(h)=\varphi(g_{[h]})=\varphi(h),$

since $h\in[h]$ & $g_{[h]}\in[h] \Rightarrow h\sim g_{[h]} \Rightarrow \varphi(h)=\varphi(g_{[h]})$, as φ is a class function

⊠

Rmk: • In fact $\mathrm{Maps}(G,\mathbb{C})$ is even an algebra over $\mathbb{C}$, with $(\varphi\psi)(g)=\varphi(g)\psi(g)$, and $CF(G)$ is its subalgebra.

• 1_C forms an orthogonal basis of $CF(G)$.

Prop. 7: χ^V, $V\in \mathrm{Irrep}(G)$, is an orthonormal basis of $CF(G)$. $(\chi^{V_i};\chi^{V_j})=\delta_{i,j}$.

This result is highly non-trivial. We postpone its proof, and instead present its deep consequences, central in the practical study of reps.

Thm 1: $\boxed{\#\mathrm{Irrep}(G)=\#\mathrm{Conj}(G)}$.

In particular, $\#\mathrm{Irrep}(G)$ is finite!

□ Compare the sizes of two bases of $CF(G)$, the one from Prop. 6 and the one from Prop. 7 ☒

Thm 2: $V\in \mathrm{Rep}(G)$, $\mathrm{Irrep}(G)=\{V_1,V_2,\dots,V_{c(G)}\}$,

$V\cong \bigoplus_{i=1}^{c(G)} m_i V_i$, $m_i\geq 0 \implies \boxed{m_i=(\chi^V;\chi^{V_i})}$

□ $(\chi^V,\chi^{V_i})=(\sum m_j\chi^{V_j},\chi^{V_i})=\sum m_j(\chi^{V_j},\chi^{V_i})=\sum m_j\delta_{i,j}=m_i$ ☒

Crl 8: The multiplicity m_i of V_i in V is well defined:

$V\cong\bigoplus m_iV_i\cong\bigoplus m_i'V_i \Rightarrow \forall i,\ m_i=m_i'$.

Rmk: • Decomposition of V into irreducible parts need not be unique.

Ex.: $n=\dim_{\mathbb{C}}V\geq 2$, $g\cdot v=v \Rightarrow \forall$ basis (e_i) of V, $V=\bigoplus_{i=1}^{n}\mathbb{C}e_i$ is a decomposition into irreps: $\mathbb{C}e_i\cong V^{tr}$.

For another basis (e_i'), one gets the same decomposition iff $\forall i$, $e_i=\lambda e_j'$ for some j & some $\lambda\in\mathbb{C}^*$.

One gets an ∞ nb of decompositions.

They correspond to decompositions of $\mathbb{C}^n$ into n lines.

• We'll see: the coarser decomposition
$V=\bigoplus_{i=1}^{c(G)} W_i$, $W_i \cong m_i V_i$, is unique.

Thm 3: $\boxed{V \cong V' \Leftrightarrow \chi^V = \chi^{V'}}$.

□ • $\underset{\rho}{V} \cong \underset{\rho'}{V'} \Rightarrow \exists \Phi: V \xrightarrow[\mathbb{C}\text{-lin.}]{\sim} V'$ s.t. 1) Φ is bijective 2) Φ is $\mathbb{C}$-linear 3) Φ is G-linear. $\Rightarrow \forall g \in G$, $\Phi \rho(g) = \rho'(g)\Phi \Rightarrow$

$\chi^{V'}(g) = tr(\rho'(g)) = tr(\Phi \rho(g) \Phi^{-1}) = tr(\Phi^{-1}\Phi \rho(g)) = tr(\rho(g)) = \chi^V(g)$.

• $\chi^V = \chi^{V'}$, $V \cong \oplus m_i V_i$, $V' \cong \oplus m_i' V_i \Rightarrow \forall i, m_i' = (\chi^V, \chi^{V_i}) = (\chi^{V'}, \chi^{V_i}) = m_i$
$\Rightarrow V \cong V'$ ⊠

So, characters characterise reps.

Thm 4 (Irreducibility criterion): For $V \in Rep(G)$,
$\boxed{V \text{ irreducible} \Leftrightarrow (\chi^V, \chi^V) = 1}$.

□ $\forall V$ writes as $\bigoplus_{i=1}^{c(G)} m_i V_i \Rightarrow (\chi^V, \chi^V) = (\sum m_i \chi^{V_i}, \sum m_j \chi^{V_j}) =$
$= \sum m_i m_j (\chi^{V_i}, \chi^{V_j}) = \sum m_i m_j \delta_{i,j} = \sum_{i=1}^{c(G)} m_i^2$,
so $(\chi^V, \chi^V) = 1 \Leftrightarrow V \cong V_i$ for some $i \Leftrightarrow V$ irreducible. ⊠

Thm 5: $\boxed{V^{reg} \cong \bigoplus_{i=1}^{c(G)} d_i V_i, \quad d_i = \dim_{\mathbb{C}}(V_i)}$.

□ $(\chi^{V^{reg}}, \chi^{V_i}) = \frac{1}{\#G} \sum_{g \in G} \chi^{V^{reg}}(g) \chi^{V_i}(g)) = \frac{1}{\#G} \sum (\#G \cdot \delta_{g,1}) \chi^{V_i}(g) =$
$= \chi^{V_i}(1) = \dim_{\mathbb{C}}(V_i)$ ⊠.

So, V^{reg} contains copies of all irreps!

Thm 6: (1) $\boxed{\sum_{i=1}^{c(G)} (\dim(V_i))^2 = \#G}$.
(2) $\boxed{\forall g \neq 1, \sum \dim(V_i) \chi^{V_i}(g) = 0}$.

□ Thm 5 $\Rightarrow \chi^{V^{reg}} = \sum d_i \chi^{V_i}$. (1) $\#G = \chi^{V^{reg}}(1) = \sum d_i \chi^{V_i}(1) = \sum d_i^2$.
(2) $0 = \chi^{V^{reg}}(g) = \sum d_i \chi^{V_i}(g)$, for $g \neq 1$. ⊠

We'll see: $\boxed{\dim(V_i) \mid \#G}$