Lecture 5: Classical functions

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MA1S11A: Calculus with Applications for Scientists

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1 Bricks and mortar

Most function you'll encounter in this module are obtained from the "elementary bricks":

- ✓ c, $c \in \mathbb{R}$ (constant functions);
- ✓ x;
- ✓ $\sin(x)$;
- ✓ e^x (exponential function)

using as "mortar" different operations:

- ✓ arithmetic operations: $f+g$, $f-g$, fg, $\frac{f}{g}$;
- ✓ powers: f^α, $\alpha \in \mathbb{R}$;
- ✓ composition: $f \circ g$;
- ✓ inverse function f^{-1};
- ✓ gluing functions from different pieces (piecewise defined functions).

We'll learn the exponential function and non-integer powers much later in the module, and inverse functions later this week.

Today we'll study the most basic functions obtained from our bricks and mortar.

2 Polynomials

Definition. A **polynomial** is a function which is a sum of finitely many power terms cx^n, where c is a real number, and n is a non-negative integer.

Examples:

$$2x-1, \quad 5x^3+x-\sqrt{3}, \quad x^2, \quad 3, \quad x^7-x^3-2,$$
$$x^{10}+\frac{\pi}{2}x^3, \quad x^2-2x^2, \quad 0.$$

It is common to collect similar terms in a polynomial. So, the last polynomial above is systematically simplified: $x^2-2x^2=-x^2$.

The function $(x-1)(x-2)^2$ is also a polynomial since we can perform the multiplication and collect similar terms:

$$(x-1)(x-2)^2=(x-1)(x^2-4x+4)=$$
$$=x^3-4x^2+4x-x^2+4x-4=x^3-5x^2+8x-4.$$

More generally, if we add, subtract, multiply, or compose polynomials, we get a polynomial.

⚠ Division does not always yield polynomials!

2 Polynomials

A general polynomial has the form

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0, \qquad c_n \neq 0,$$

which is the same as $c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + c_n x^n$.

The numbers $c_0, c_1, \ldots, c_n$ are called the **coefficients** of the polynomial.

The highest power n occurring with a non-zero coefficient is called the **degree** of the polynomial.

A polynomial of degree 0 is a constant $c = cx^0$. We declare 0 to be a polynomial of undefined degree.

Polynomials of degrees $1, 2, 3, 4, 5$ are referred to as *linear*, *quadratic*, *cubic*, *quartic*, *quintic* polynomials respectively.

A polynomial of the form $c_n x^n$ is called a **monomial**.

A **polynomial function** is a function of the form

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0.$$

2 Polynomials

Polynomial are obtained from two types of "elementary bricks" only:

- ✓ c, $c \in \mathbb{R}$ (constant functions);
- ✓ x;

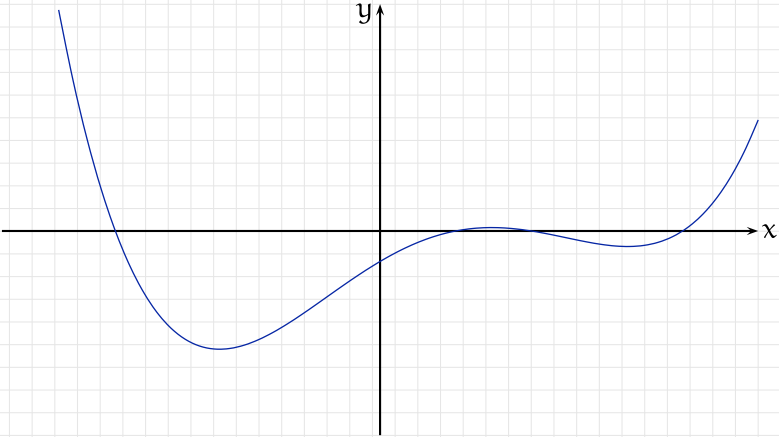
using the simplest "mortar":

- ✓ addition $f + g$;
- ✓ multiplication fg.

2 Polynomials

Examples of graphs of polynomial functions of degree 4:

$$f(x) = (x-2)(x-4)(x-8)(x+7).$$



Observe that this graph has no symmetries!

2 Polynomials

Examples of graphs of polynomial functions of degree 4 (zoomed out):

$$f(x) = (x-2)(x-4)(x-8)(x+7), \qquad f(x) = x^4.$$

2 Polynomials

For large $|x|$ there is little difference between the graphs of polynomial functions $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ and just $c_n x^n$ (we suppose $c_n \neq 0$).

Indeed, when $k < n$, and when $|x|$ gets arbitrarily large, $|x^{n-k}| = |x|^{n-k} \geqslant |x|$ gets arbitrarily large, so

$$\frac{c_k x^k}{c_n x^n} = \frac{c_k}{c_n} \frac{1}{x^{n-k}}$$

gets arbitrarily small. Thus, lower terms, when compared to the term with the highest degree, get negligible.

We've just seen a typical calculus argument!

3 Rational functions

Definition. A **rational function** is a ratio of two polynomials, $\frac{P(x)}{Q(x)}$.

Examples: $\frac{1}{x^2+1}$, $\frac{x^2+2x}{x^2-1}$, $\frac{x^2+1}{x}$.

Rational functions are obtained from two types of "elementary bricks" only:

✓ c, $c \in \mathbb{R}$ (constant functions);

✓ x;

using as "mortar"

✓ addition $f+g$;

✓ multiplication fg;

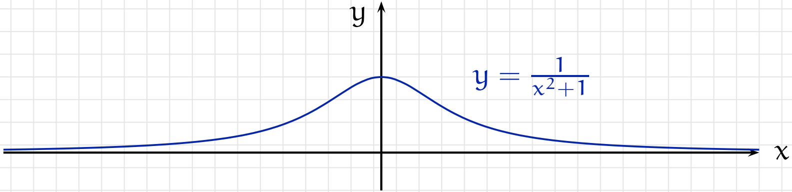
✓ division $\frac{f}{g}$.

3 Rational functions

Definition. A **rational function** is a ratio of two polynomials, $\frac{P(x)}{Q(x)}$.

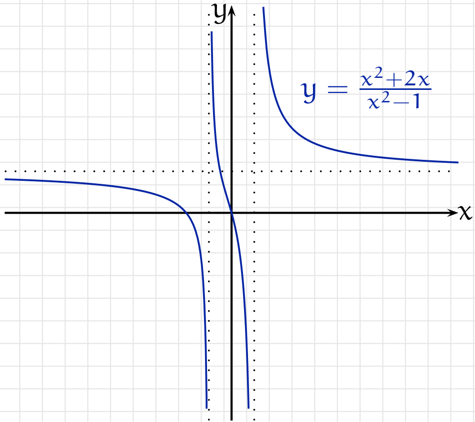
While the natural domain of polynomials was $\mathbb{R}$, rational functions are undefined where the denominator vanishes ($Q(x) = 0$). In such cases, the corresponding graphs have **vertical asymptotes** (vertical lines that they closely approximate).

Rational functions might also have **horizontal asymptotes**, although that's not always the case.



The graph of $f(x) = \frac{1}{x^2+1}$ has a horizontal asymptote $y = 0$, and no vertical asymptotes, since the natural domain of f is $\mathbb{R}$. Indeed, $1 + x^2 > 0$ for all real x.

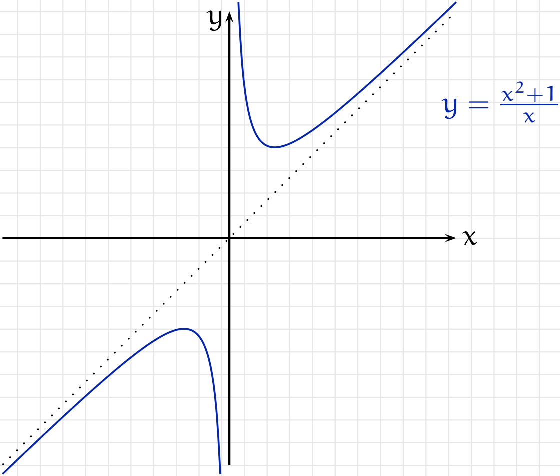
3 Rational functions



The graph of $\frac{x^2+2x}{x^2-1}$ has vertical asymptotes $x = 1$ and $x = -1$ (values at which $x^2 - 1 = 0$), and a horizontal asymptote $y = 1$. To see the latter, write

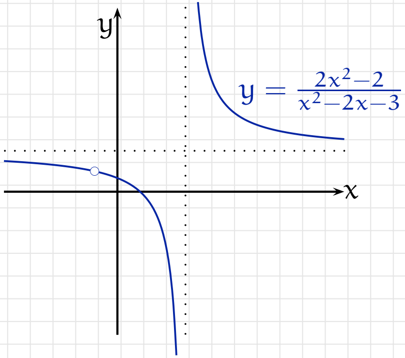
$$\frac{x^2+2x}{x^2-1} = \frac{x^2-1+1+2x}{x^2-1} = \frac{x^2-1}{x^2-1} + \frac{2x+1}{x^2-1} = 1 + \frac{2x+1}{x^2-1}.$$

3 Rational functions



This graph has a vertical asymptote $x = 0$, and no horizontal asymptotes. It however has another straight line asymptote $y = x$, because $\frac{x^2+1}{x} = x + \frac{1}{x}$.

3 Rational functions



This graph has a vertical asymptote $x = 3$, a horizontal asymptote $y = 2$, and also the point $(-1, 1)$ which it approaches both on the left and on the right but does not touch. Indeed, for $x \neq -1$, we have

$$\frac{2x^2-2}{x^2-2x-3} = \frac{2(x-1)(x+1)}{(x+1)(x-3)} = \frac{2x-2}{x-3} = \frac{2(x-3)+4}{x-3} = 2 + \frac{4}{x-3}.$$

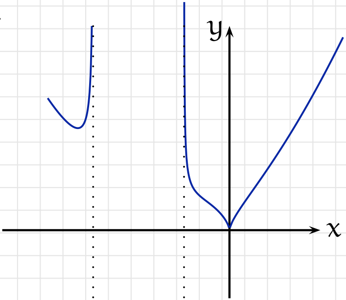
So, our graph is obtained from that of $f(x) = \frac{1}{x}$ by a vertical scaling ($\frac{4}{x}$), horizontal ($\frac{4}{x-3}$) and vertical ($2 + \frac{4}{x-3}$) shifts, and a cut out at $x = 1$.

4 Algebraic functions

Definition. **Algebraic functions** are built from the constants and the variable x using the four arithmetic operations, and extracting roots.

Examples: $\dfrac{\sqrt{x}-2x^3}{x^2+1}$, $\dfrac{x^2+2x\sqrt{x-\frac{6}{x+2}}}{\sqrt[3]{x^2-1}+\pi}$, $\dfrac{x^2+1}{x+\frac{x}{1+\sqrt{1+\sqrt{x}}}}$.

Algebraic functions are too varied in behaviour to make any general statements. They are only defined for some values, their graphs may have corners etc.



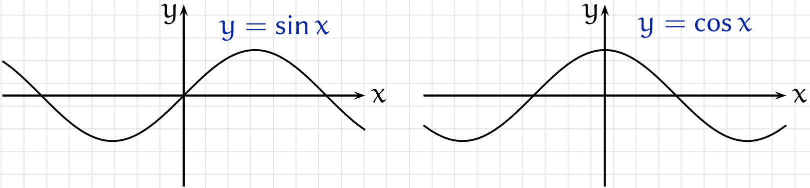
$$y = \frac{\sqrt[3]{x^2}(x+2)^2}{\sqrt{x^2+4x+3}}$$

5 Trigonometric functions

All functions you see in trigonometry can be obtained from $\sin(x)$ using the four arithmetic operations, and composition with linear functions ($ax + b$).

Examples: $\cos(x) = \sin(x + \frac{\pi}{2})$, $\quad \tan(x) = \frac{\sin(x)}{\cos(x)}$, $\quad \cot(x) = \frac{\cos(x)}{\sin(x)}$.

Here we will mainly talk about sin and cos:



function	nat. domain	range	odd/even	periodic
sin	$\mathbb{R}$	$[-1, 1]$	odd	yes, with period 2π
cos	$\mathbb{R}$	$[-1, 1]$	even	yes, with period 2π

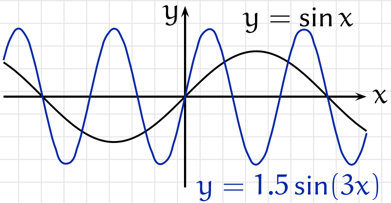
By the way, negative angles do make perfect sense: we often need to distinguish between the clockwise direction (usually considered *negative*) and the counterclockwise direction (*positive*).

5 Trigonometric functions

If we look at the function $g(x) = A\sin(Bx)$, $B \neq 0$, we observe that it is $\frac{2\pi}{B}$-periodic:

$$A\sin(B(x + \frac{2\pi}{B})) = A\sin(Bx + 2\pi) = A\sin(Bx).$$

Also, while sin oscillates between -1 and 1, this new function oscillates between $-A$ and A.



The same applies to the function $A\cos(Bx)$, and also to the functions $A\sin(Bx - C) = A\sin(B(x - \frac{C}{B}))$ and $A\cos(Bx - C) = A\cos(B(x - \frac{C}{B}))$.

5 Trigonometric functions

Besides the period $T = \frac{2\pi}{B}$ and the amplitude A, a quantity used in applications of trigonometric functions is the frequency $\frac{1}{T} = \frac{B}{2\pi}$, number of periods passed in one unit of time.

If A or B is negative, we should use absolute values in these formulas and write $\frac{2\pi}{|B|}$ for the period, $|A|$ for the amplitude, and $\frac{|B|}{2\pi}$ for the frequency.

5 Trigonometric functions

Linguistic digression:

Where does the word *sine* comes from?

From Latin *sinus*, meaning “bend”, “bay”, or “the bosom of a garment”,more specifically “the hanging fold of the upper part of a toga”

5 Trigonometric functions

Linguistic digression:
Where does the word *sine* comes from?
From Latin *sinus*, meaning “bend”, “bay”, or “the bosom of a garment”, more specifically “the hanging fold of the upper part of a toga”. You might know the word from its anatomical meaning: the cavities or bays in the facial bones.
The word *sinus* was chosen as the translation of what was mis-interpreted as the Arabic word meaning “pocket” or “fold” in the 12th century. A possible reason of the mis-interpretation is the vowel omission in the Arabic language. The original Arabic name meant “chord”, “bowstring”. It was a neologism borrowed from the Hindu (where it has been used since 500AD).