4. Tracing Representations: Characters

So far we have learned that a rep. of a finite group decomposes into irreducible summands. Today we will introduce a key tool for carrying out this decomposition in practice.

Def.: The **character** of $(V, \rho) \in Rep(\mathcal{G})$ is the map

$$\chi^{(V,\rho)}: \mathcal{G} \to \mathbb{C},$$
$$g \mapsto tr(\rho(g)).$$

In practice, one computes the trace (= sum of diagonal elements) of the matrix of $\rho(g) \in Aut_{\mathbb{C}}(V)$ in a suitable basis of V.

⚠ In general, $\chi^{(V,\rho)}$ is not a group morphism:

- $\chi(g_1 g_2) \neq \chi(g_1)\chi(g_2)$ for certain g_1, g_2;
- even worse: $\chi(g) = 0$ for some g!

Characters can be considered within 2 frameworks:

(1) "approximation" by simpler objects:

$$\text{a group} \xrightarrow[\text{by matrices}]{\text{replace el-nts}} \text{reps} \xrightarrow[\text{by their traces}]{\text{replace matrices}} \text{chars}$$

(2) duality: to understand a $\underset{\mathcal{G}}{\underline{\text{group}}}$, study $\underset{\mathcal{G} \to \mathbb{C}}{\underline{\text{functions}}}$ on it.

Why did we choose trace, and not determinant or another map $Aut_{\mathbb{C}}(V) \to \mathbb{C}$?

Because $\chi^{(V,\rho)}$ captures essential information on (V, ρ)!

Spoiler: For a finite group $\mathcal{G}$, put

$$(\varphi, \psi) = \frac{1}{\# \mathcal{G}} \sum_{g \in \mathcal{G}} \varphi(g) \overline{\psi(g)} \quad \in \mathbb{C}$$

for all $\varphi, \psi : \mathcal{G} \to \mathbb{C}$. Then

(1) For $V_1, V_2 \in Rep(\mathcal{G})$, $\boxed{V_1 \underset{\substack{\uparrow \\ \text{isomorphic} \\ \mathcal{G}\text{-reps}}}{\cong} V_2 \Leftrightarrow \chi^{V_1} = \chi^{V_2}}$.

So characters characterise reps, hence the name.

(2) $V \in Rep(\mathcal{G})$: $\boxed{V \text{ is irred.} \Leftrightarrow (\chi^V, \chi^V) = 1}$ (Irreducibility criterion)

(3) $V \cong m_1 V_1 \oplus \ldots \oplus m_k V_k$, where
as $\mathcal{G}$-reps
- $V \in Rep(\mathcal{G})$
- $V_i \in Irrep(\mathcal{G})$
- $V_i \not\cong V_j$ for $i \neq j$
- $m_i V_i$ stands for $\underbrace{V_i \oplus \ldots \oplus V_i}_{m_i \text{ summands}}$

$\Downarrow$

$\boxed{m_i = (\chi^V, \chi^{V_i})}$

multiplicity of V_i in V.

One gets simple numerical methods for
(1) comparing reps;
(2) testing irreducibility;
(3) decomposing reps into irreps.

Let us return to basics on characters.

Ex.: (1) $\mathcal{G}$ finite, $V^{reg} = \mathbb{C}\mathcal{G}$ is the left regular rep.: $g \cdot e_h = e_{gh}$.

$\boxed{\chi^{V^{reg}}(g) = \# \mathcal{G}\, \delta_{g,1}}$ Here $\delta_{g,1} = \begin{cases} 1, & g = 1 \\ 0, & g \neq 1 \end{cases}$.

Indeed, in the basis $(e_h)_{h \in \mathcal{G}}$, the matrix of $\rho^{reg}(g)$ is $\begin{array}{cc} & h \\ & \downarrow \\ & \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ \cdots & 1 & \cdots \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix} & \leftarrow gh \end{array}$

So $\chi^{V^{reg}}(g) = tr(\rho^{reg}(g)) = \sum_{h \in \mathcal{G} | h = gh} 1 = \begin{cases} \# \mathcal{G}, & \text{if } g = 1, \\ 0, & \text{if } g \neq 1. \end{cases}$

(2) More generally, for a permutation rep. $\mathbb{C}\mathcal{X}$, where $\mathcal{X}$ is a finite $\mathcal{G}$-set and $g \cdot e_x = e_{g \cdot x}$ $\forall x \in \mathcal{X}, g \in \mathcal{G}$, one has

$\boxed{\chi^{\mathbb{C}\mathcal{X}}(g) = \# \mathcal{X}^g}$ Here $\mathcal{X}^g = \{x \in \mathcal{X} \mid g \cdot x = x\}$, the set of fixed points for g in $\mathcal{X}$.

The proof is similar to (1), and uses the matrix of $\rho^{\mathbb{C}\mathcal{X}}(g)$. $\begin{array}{cc} & x \\ & \swarrow \\ & \begin{pmatrix} & 0 & \\ & \vdots & \\ \cdots & 1 & \cdots \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix} & \leftarrow g \cdot x \end{array}$

In this example, the character $\chi^{\mathbb{C}X'}$ encodes important combinatorial information about the G-set X'.

Prop. 5: Let $V, V' \in \mathrm{Rep}(G)$, $g, h \in G$.

a) $\chi^V(1) = \dim_{\mathbb{C}}(V) =: n$ $\Rightarrow$ χ^V "knows" at least $\dim_{\mathbb{C}} n$.

b) $\chi^V(hgh^{-1}) = \chi^V(g)$

for finite G:

c) $\chi^V(g)$ is a sum of n roots of 1

d) $\chi^V(g^{-1}) = \overline{\chi^V(g)}$ complex conjugate

$\Rightarrow$ accelerates the computation of χ^V.

e) $\chi^{\{0\}}(g) = 0$

f) $\chi^{V\oplus V'}(g) = \chi^V(g) + \chi^{V'}(g)$

$\Rightarrow$ $(\mathrm{Rep}(G), \oplus, \{0\}) \xrightarrow{\chi} \mathrm{Maps}(G, \mathbb{C})$, $V \longmapsto \chi^V$ is a monoid morphism.

$\square$ a) $\chi^V(1) = tr(\rho(1)) = tr(Id_V) = tr\underbrace{\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}}_{n} = n.$

b) $\chi^V(hgh^{-1}) = tr(\rho(hgh^{-1})) = tr(\rho(h)\rho(g)\rho(h)^{-1}) \overset{\checkmark}{=}$ basic property of the trace: $tr(ab) = tr(ba)$

$= tr(\rho(h)^{-1}\rho(h)\rho(g)) = tr(\rho(g)) = \chi^V(g).$

c) Jordan normal form $\Rightarrow$ In a good basis, the matrix $M(g)$ of $\rho(g)$ writes
$$\left(\begin{array}{ccc|c|cc} \lambda_1 & 1 & 0 & & & \\ & \lambda_1 & 1 & & 0 & \\ 0 & & \lambda_1 & & & \\ \hline & & & \lambda_2 & & \\ \hline & 0 & & & \lambda_3 & 1 \\ & & & & 0 & \lambda_3 \\ & & & & & & \ddots \end{array}\right)$$
for some $\lambda_i \in \mathbb{C}$ and blocks of some size.

G finite $\Rightarrow g^d = 1$ for some $d \Rightarrow M(g)^d = \begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$.

The powers of Jordan blocks look as follows:

$(\lambda)^k = (\lambda^k)$, $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^k = \begin{pmatrix}\lambda^k & k\lambda^{k-1}\\ 0 & \lambda^k\end{pmatrix}$, $\begin{pmatrix}\lambda & 1 & 0\\ & \lambda & 1\\ 0 & & \lambda\end{pmatrix}^k = \begin{pmatrix}\lambda^k & k\lambda^{k-1} & *\\ & \lambda^k & k\lambda^{k-1}\\ 0 & & \lambda^k\end{pmatrix}$, etc.

So $M(g)^d = \begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix} \Rightarrow$ all blocks of $M(g)$ are of size 1, with $\lambda_i^d = 1$. But then $\chi^V(g) = tr(M(g)) = \sum_{i=1}^{n} \lambda_i$, and each λ_i is a d^{th} root of 1.

d) By c), $\chi^V(g^{-1}) = tr(M(g)^{-1}) = tr\begin{pmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{pmatrix} = \sum_{i=1}^n \lambda_i^{-1} = \sum_{i=1}^n \overline{\lambda_i} =$

$= \overline{\sum_{i=1}^n \lambda_i} = \overline{\chi^V(g)}.$

We used $\lambda_i^{-1} = \overline{\lambda_i}/|\lambda_i|^2 = \overline{\lambda_i}$, since $|\lambda_i| = 1 \Leftarrow \lambda_i^d = 1$.

e) Obvious.

f) $\chi^{V\oplus V'}(g) = tr(\rho^{V\oplus V'}(g)) = tr(\rho^V(g) \oplus \rho^{V'}(g)) = tr(\rho^V(g)) + tr(\rho^{V'}(g))$

$= \chi^V(g) + \chi^{V'}(g).$

Alternatively, choose bases $\mathcal{B}$ & $\mathcal{B}'$ of V & V'. Then $\mathcal{B} \sqcup \mathcal{B}'$ is a basis of $V \oplus V'$. Let $M^V(g)$, $M^{V'}(g)$ & $M^{V\oplus V'}(g)$ be the matrices of $\rho(g)$, $\rho'(g)$ and $(\rho\oplus\rho')(g)$ in these bases. Then

$$M^{V\oplus V'}(g) = \begin{pmatrix} M^V(g) & 0 \\ 0 & M^{V'}(g) \end{pmatrix}.$$

$\boxtimes$

Rmk: For finite G, we have shown (in the proof of c) that $\chi^{(V,\rho)}(g) = \sum(\text{eigenvalues } \lambda_i \text{ of } \rho(g))$.

Then $\chi^{(V,\rho)}(g^k) = \sum_{i=1}^n \lambda_i^k$. By the symmetric function theory, the knowledge of $\sum \lambda_i^k$ for all k allows you to extract the λ_i. So $\chi^{(V,\rho)}$ "knows" the eigenvalues of all $\rho(g)$, which are the key characteristics of linear automorphisms. This explains why chars are so efficient in the study of reps.