Lecture 4: Functions with symmetries. Parametric families of functions

Victoria LEBED, lebed@maths.tcd.ie

MA1S11A: Calculus with Applications for Scientists

October 9, 2017

1 Symmetries in sciences

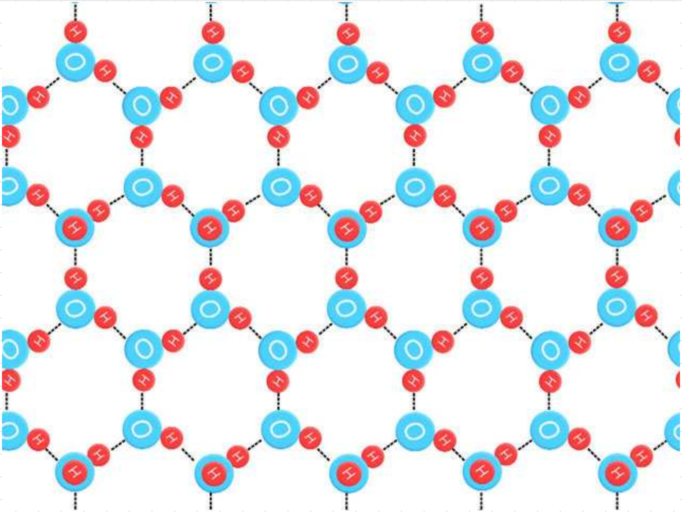
In all sciences, you inevitably meet

- ✓ transformations of various systems you work with;
- ✓ symmetric systems, i.e. those unchanged by the corresponding transformations.

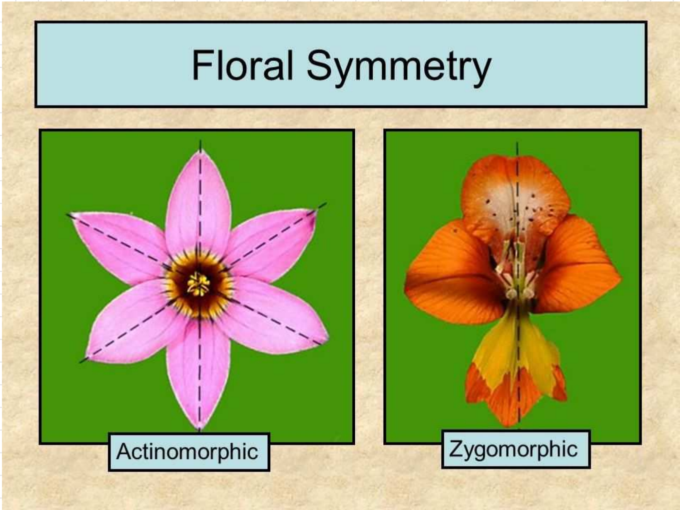
Here the word “symmetric” is employed in a wider sense that you are probably used to.

You encounter symmetries in physics:

You encounter symmetries in chemistry:



You encounter symmetries in biology:



The good thing about something symmetric is that understanding a part of it suffices for understanding the whole object.

2 Symmetries in calculus

In calculus, our main objects of study is functions.

We have already seen their major transformations:

function transformation	graph transformation
$f(x) \rightsquigarrow f(x+a)$	horizontal shift
$f(x) \rightsquigarrow f(x)+a$	vertical shift
$f(x) \rightsquigarrow f(-x)$	horizontal reflection (about the y-axis)
$f(x) \rightsquigarrow -f(x)$	vertical reflection (about the x-axis)
$f(x) \rightsquigarrow f(cx)$	horizontal compression/stretching
$f(x) \rightsquigarrow cf(x)$	vertical compression/stretching

$c > 1$: vertical stretching, horizontal compression;
$0 < c < 1$: vertical compression, horizontal stretching.

Today we will talk about symmetries with respect to these transformations.

3 Functions with symmetries

Definition. A function f is said to be

- ✓ **even** if for all x in the domain of f, $-x$ is also in the domain of f, and $f(-x) = f(x)$;
- ✓ **odd** if for all x in the domain of f, $-x$ is also in the domain of f, and $f(-x) = -f(x)$;
- ✓ **periodic with period** T if for all x in the domain of f, $x + T$ is also in the domain of f, and $f(x + T) = f(x)$.

⚠ Don't forget to check that $-x$ or $x + T$ is in the domain of f whenever x is! Students often omit this impotent step.

Example. The function $f(x) = x$ for $x \geqslant 1$ is not odd, since f is defined at $x = 1$, but not at $x = -1$.

3 Functions with symmetries

function property	graph property
even	symmetric about the y-axis
odd	symmetric about the origin $(0,0)$
periodic with period T	unchanged under the horizontal shift by T units

Being aware of symmetries allows us to study a function just on a part of its domain (only for $x \geqslant 0$, or on $[a, a+T)$), and derive information elsewhere by using symmetry.

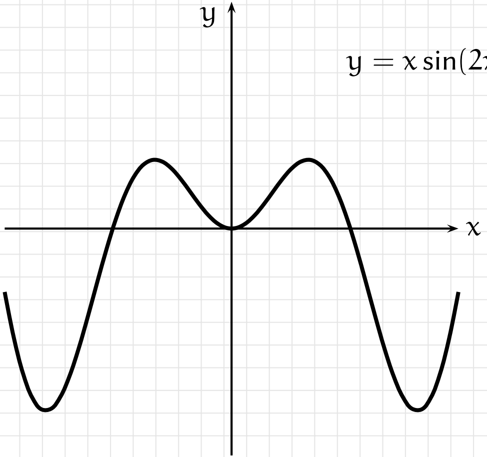
One more application of symmetry:
The average value (hence, the integral) of an odd function is 0, since $f(x) + f(-x) = 0$ for all x in its domain.

4 Even functions

Let us take the function $f(x) = x\sin(2x)$. Since

$$f(-x) = (-x)\sin(-2x) = (-x)(-\sin(2x)) = x\sin(2x),$$

this function is even. The corresponding graph looks as follows:

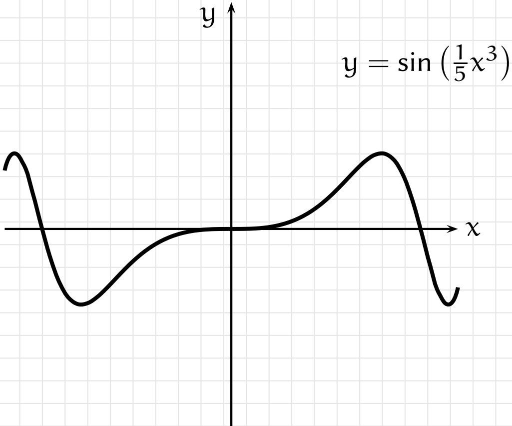


5 Odd functions

Let us take the function $f(x) = \sin\left(\frac{1}{5}x^3\right)$. Since

$$f(-x) = \sin\left(\frac{1}{5}(-x)^3\right) = \sin\left(-\frac{1}{5}x^3\right) = -\sin\left(\frac{1}{5}x^3\right),$$

this function is odd. The corresponding graph looks as follows:

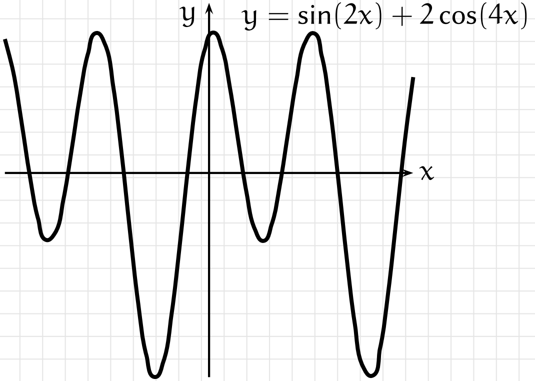


6 Periodic functions

Let us take the function $f(x) = \sin(2x) + 2\cos(4x)$. Since

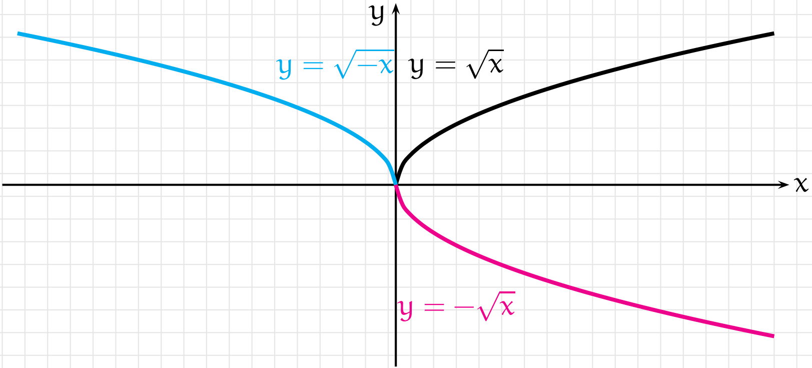
$$f(x+\pi) = \sin(2(x+\pi)) + 2\cos(4(x+\pi)) =$$
$$= \sin(2x+2\pi) + 2\cos(4x+4\pi) = \sin(2x) + 2\cos(4x) = f(x),$$

this function is periodic with period π.



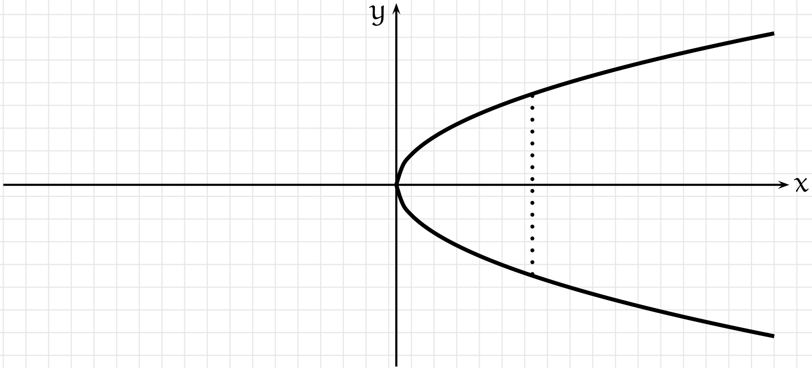
7 Symmetry about the x-axis

What about the symmetry about the x-axis? We discussed that type of symmetry the last time:



7 Symmetry about the x-axis

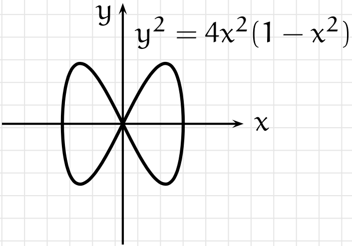
A graph of a function cannot be symmetric about the x-axis, since that would break the vertical line test:

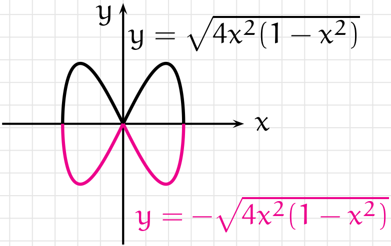


The only exception is the graph of the zero function $f(x) = 0$, since in this case each vertical line meets the graph at the x-axis, so there is just one intersection point.

7 Symmetry about the x-axis

It can however be useful to apply the symmetry about the x-axis to plot curves defined by equations:



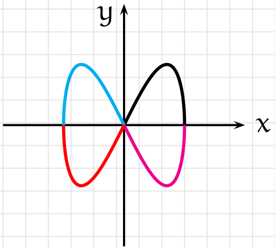


8 Symmetries for curves

For curves defined by equations we have the following symmetry tests:

- ✓ A plane curve is symmetric about the y-axis iff (= if and only if) its equation does not change under replacing x by $-x$;
- ✓ A plane curve is symmetric about the x-axis iff its equation does not change under replacing y by $-y$;
- ✓ A plane curve is symmetric about the origin iff its equation does not change under replacing x by $-x$ and y by $-y$ simultaneously.

Example. The curve $y^2 = 4x^2(1 - x^2)$ satisfies all these conditions, so it has many different symmetries:



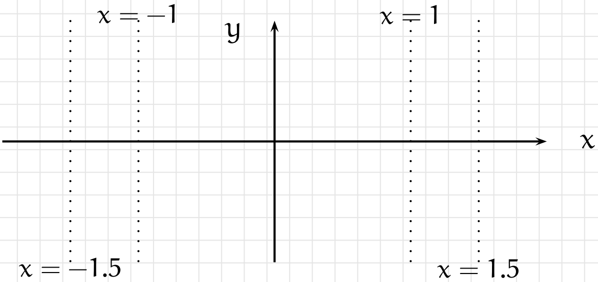
9 Families of straight lines

Let us, alongside with building a "vocabulary" for talking about functions, start building a "library" of functions.

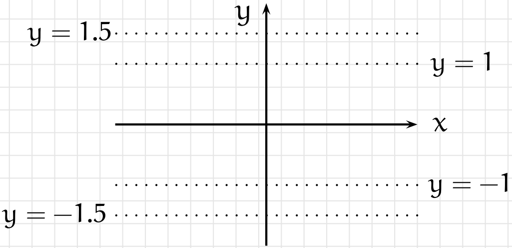
Geometrically, the simplest possible shape is the straight line:

- ✓ horizontal straight lines are defined by equations $y = c$ for various c;
- ✓ vertical lines are defined by equations $x = c$ for various c;
- ✓ general non-vertical lines are defined by equations $y = mx + b$ for various m (slopes) and b (shifts);
- ✓ general lines are defined by linear equations $ax + by = c$ for various a, b, c.

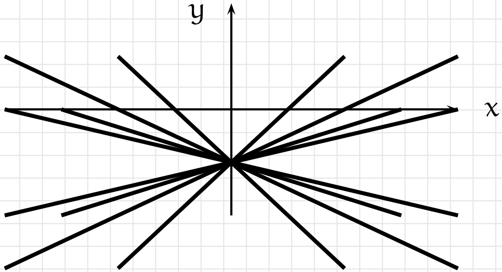
9 Families of straight lines



9 Families of straight lines

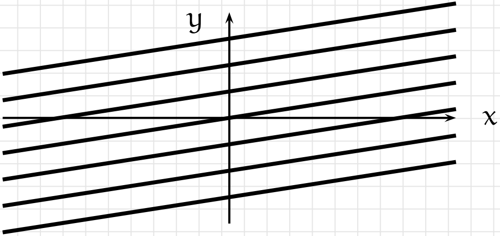


9 Families of straight lines



These "whiskers" are lines $y = mx - 1$ for various m. Note that for $x = 0$ the formula produces -1 regardless of what m is. So, all these lines intersect at the point $(0, -1)$.

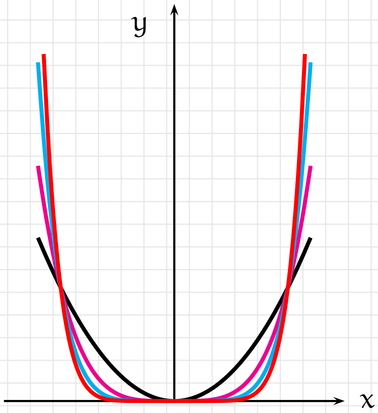
9 Families of straight lines



These lines are $y = x/3 + b$ for various b. Note that they all are indeed obtained from $y = x/3$ by vertical shifts.

10 Families of power functions

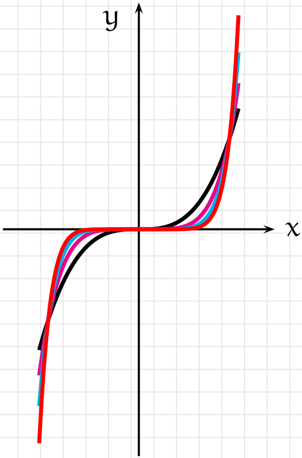
Another basic class of functions is the power functions, $y = x^n$. We shall consider several different options for n, assuming that it is an integer.



This illustrates the behaviour of $y = x^n$ for even $n = 2, 4, 6, 8$.

These functions are even, and intersect at three points: $(-1, 1)$, $(0, 0)$, $(1, 1)$.

10 Families of power functions

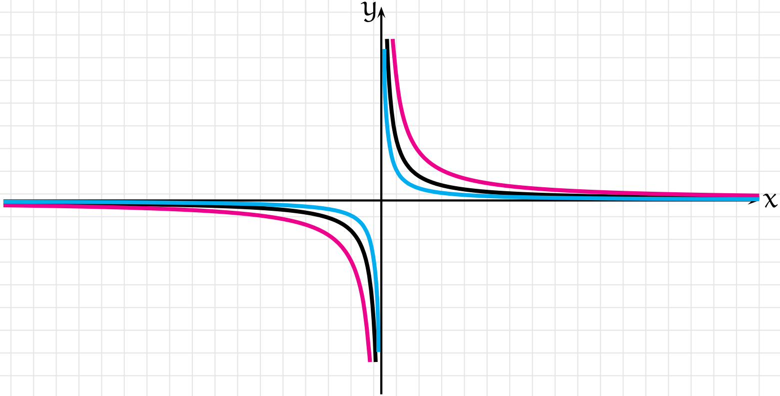


This illustrates the behaviour of $y = x^n$ for odd $n = 3, 5, 7, 9$.

These functions are odd, and intersect at three points: $(-1, -1)$, $(0, 0)$, $(1, 1)$.

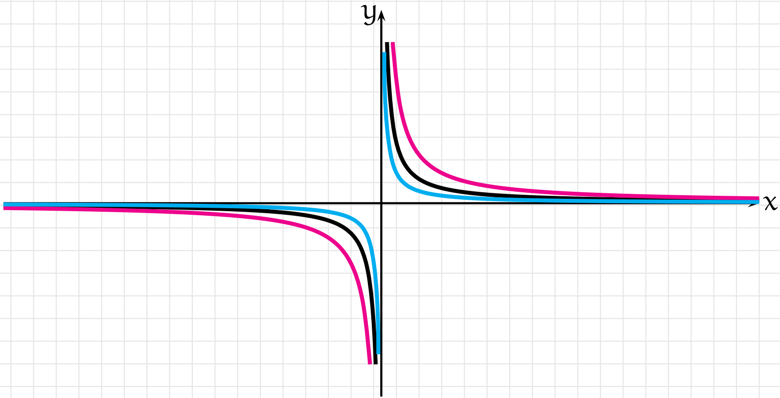
11 Families $y = k/x$

The inverse proportionality $y = k/x$ is a function that appears in many situations, e.g. Boyle's Law $PV = k$ (for a fixed amount of an ideal gas at constant temperature).



This illustrates the behaviour of $y = k/x$ for $k = 1/2, 1, 2$. The larger k, the further the graph would be from the origin.

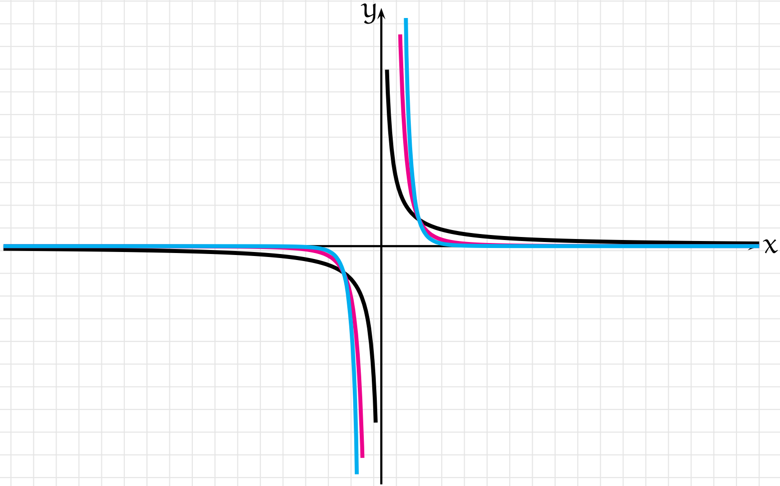
11 Families $y = k/x$



This illustrates the behaviour of $y = k/x$ for $k = 1/2, 1, 2$. The larger k, the further the graph would be from the origin.

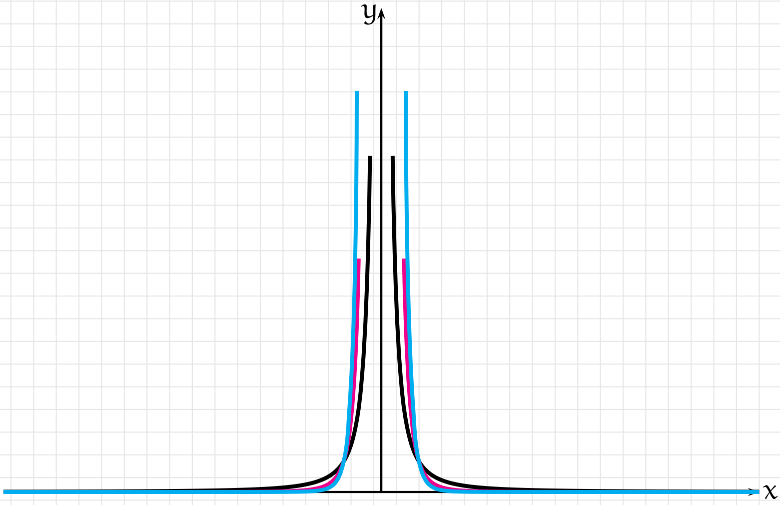
The natural domain and the range of all these functions are $(\infty, 0) \cup (0, +\infty)$. They are all odd, and do not intersect.

12 Families of inverse power functions



This illustrates the behaviour of $y = x^n$ for odd $n = -1, -3, -5$. These functions are odd, and intersect at two points: $(-1, -1)$ and $(1, 1)$.

12 Families of inverse power functions



This illustrates the behaviour of $y = x^n$ for even $n = -2, -4, -6$. These functions are even, and intersect at two points: $(-1, 1)$ and $(1, 1)$.

13 Summary for power functions

- ✓ for even $n \geqslant 2$ we get something that looks like the parabola (but is flatter close to the origin, and steeper far from the origin);
- ✓ for odd $n \geqslant 3$ we get something that looks like the parabola for $x > 0$ (but is flatter close to the origin, and steeper far from the origin), and is obtained by a reflection about the x-axis for $x < 0$;
- ✓ for odd $n \leqslant -1$, we get something that looks like the graph of inverse proportionality $y = k/x$ (but steeper close to the origin, and flatter far from the origin);
- ✓ for even $n \leqslant -2$, we get something that looks like the graph of inverse proportionality $y = k/x$ for $x > 0$ (but steeper close to the origin, and flatter far from the origin), and is obtained by a reflection about the x-axis for $x < 0$;
- ✓ for non-integer exponents, graphs are similar to those for integer ones; we shall discuss that in more detail a bit later.