3. Building Blocks: Irreducible Representations

Today we will see how general the phenomena observed in the example of S_3-reps (Lecture 2) are.

Aim: Understand Rep(G)
$$=$$
$$\{\rho : G \xrightarrow[\text{grp}]{} \mathrm{Aut}_{\mathbb{C}}(V) \mid V \text{ f.-d. vector space over } \mathbb{C}\} / \text{iso}$$

up to isomorphism (to be explained)

The S_3 example suggests a strategy: decompose reps into elementary pieces.

Def: $V \in \mathrm{Rep}(G)$ is **irreducible** (=simple) if it has precisely two sub-reps: V itself & $\{0\}$. $\mathrm{Irrep}(G) = \{\text{irreps of } G\}/\text{iso}$

irreducible reps

Rmk: $\{0\}$ is not irreducible!

Def: A **sub-representation** of $V \in \mathrm{Rep}(G)$ is its G-invariant sub-space $V' \subseteq V$.

$\forall g \in G,\ g \cdot V' = V'$

Spoiler: For a finite G,

- $\mathrm{Irrep}(G) = \{V_1, V_2, \ldots, V_{\underbrace{K(G)}_{\text{finite}}}\}$,
- $\forall\, V \in \mathrm{Rep}(G)$, $V \simeq \underbrace{m_1 V_1}_{= \underbrace{V_1 \oplus \cdots \oplus V_1}_{m_1 \text{ times}}} \oplus\, m_2 V_2 \oplus \cdots \oplus m_K V_K$,

with uniquely defined multiplicities $m_i \in \mathbb{N} \cup \{0\}$.

Compare with the unique prime factorisation thm in arithmetics.

So, $\boxed{\begin{array}{c}\text{understand Rep}(G) \\ = \\ \text{describe Irrep}(G) \\ + \\ \text{learn to decompose}\end{array}}$

describe Irrep(G) ⟵ $K(G) = ?$ $\dim_{\mathbb{C}}(V_i) = ?$

learn to decompose ⟵ $m_i = ?$

Def.: (Homo-)morphisms and isomorphisms between G-reps (V,ρ) and (V',ρ'):

- $\mathrm{Hom}_G(V,V') = \{\varphi : V \to V' \mid \varphi$ is $\mathbb{C}$-linear & G-linear $(= G$-equivariant$)\}$

  G-linear: $\forall g \in G,\ \varphi\rho(g) = \rho'(g)\varphi$ (*)

  $(\Leftrightarrow \forall g \in G, v \in V,\ \varphi(g\cdot v) = g\cdot\varphi(v))$

- $\mathrm{Iso}_G(V,V') = \{$bijective $\varphi \in \mathrm{Hom}_G(V,V')\}$

- $V \cong V'$ if $\mathrm{Iso}_G(V,V') \neq \emptyset$.

  $\cong$: isomorphic

Such definitions are standard in maths: (iso-)morphisms between two objects are (bijective) maps preserving all the structure. Here "structure" might be: group, vector space, topological object, or, in our case, G-rep.

Isomorphic G-reps should be thought of as "the same up to a change of basis" ((*) is precisely the change of basis f-la from linear algebra!). So it is natural to identify them when trying to describe all G-reps.

Prop. 1: • $\forall V, V' \in \mathrm{Rep}(G)$, $V \oplus V'$ (the direct sum of vector spaces) is also a G-rep., with

$$g\cdot(v,v') = (g\cdot v, g\cdot v');$$

• $(\mathrm{Rep}(G), \oplus, \{0\})$ is a commutative monoid. ($\{0\}$: zero rep.)

Exo: Prove.

The construction above is called the direct sum of G-reps. (V & V' are called direct summands of $V \oplus V'$)

The symbol "$\oplus$" may refer to a direct sum of spaces or of G-reps; when not clear from the context, its meaning will be given explicitly.

Mashke thm (1898): Let (V,ρ) be a rep. of a finite group G. Then any sub-rep. $V'\subseteq V$ admits a G-invariant complement V'': i.e., $V=V'\oplus V''$ (as G-reps).

$\square$ Linear algebra: $V=V'\oplus\hat{V}$ as vector spaces, for some subspace $\hat{V}$. In general, $\hat{V}$ needs not be G-invariant! We will modify it so that it becomes so.

Consider the projection $p: V=V'\oplus\hat{V}\longrightarrow V'$.

$$(v',\hat{v})\longmapsto v'$$

Put $\underset{\text{averaged}}{p^{av}}=\frac{1}{\#G}\sum_{g\in G}\rho(g)\,p\,\rho(g)^{-1}$.

⚠ Averaging is a key tool in the rep. theory of finite groups. For (locally) compact groups, it is replaced with an integral

It still projects V onto V':

(a) p^{av} is $\mathbb{C}$-linear

(b) $p^{av}(V)\subseteq V' \Leftarrow \rho(g)\underbrace{p\,\rho(g)^{-1}(v)}_{\in V'}\in V'$

(c) $\forall v'\in V'$, $p^{av}(v')=v' \Leftarrow \rho(g)\,p\,\rho(g)^{-1}(v')=\rho(g)\,p\,\underbrace{\rho(g^{-1})(v')}_{\in V'}=$

$=\rho(g)\rho(g^{-1})(v')=v'$.

So $V=\underbrace{p^{av}(V)}_{\shortparallel\, V'}\oplus\underbrace{(Id_V-p^{av})(V)}_{\shortparallel\, V''}$ as vector spaces.

Let us prove this decomposition (true for any projection!)

(1) $\forall v\in V$, $v=p^{av}(v)+(Id_V-p^{av})(v)$, so the sum of V' and V'' covers the whole V.

(2) $v\in V'\cap V'' \underset{\text{by (c)}}{\Rightarrow} p^{av}(v)=v$, but $v\in V''\Rightarrow v=\hat{v}-p^{av}(\hat{v})$ for some $\hat{v}\Rightarrow v=p^{av}(v)=p^{av}(\hat{v})-p^{av}(\underbrace{p^{av}(\hat{v})}_{\in V'})=$

$=p^{av}(\hat{v})-p^{av}(\hat{v})=0$.

So $V'\cap V''=\{0\}$, i.e., V' and V'' are in direct sum.

We now show that, contrary to $\tilde{V}$, V'' is G-invariant.
First, observe that $p^{av} \in \mathrm{Hom}_G(V, V')$. Indeed,

$$p^{av}\rho(h) = \frac{1}{\# G}\sum_{g\in G}\rho(g)\,p\,\underbrace{\rho(g)^{-1}\rho(h)}_{=\rho(g^{-1})\rho(h)=\rho(g^{-1}h)} = \frac{1}{\# G}\sum_{g\in G}\rho(g)\,p\,\rho(g^{-1}h)$$

$$\underset{\substack{\text{change}\\ \text{of variables: } g'=h^{-1}g}}{=} \frac{1}{\# G}\sum_{g'\in G}\rho(hg')\,p\,\rho((g')^{-1}) = \frac{1}{\# G}\sum_{g'\in G}\rho(h)\rho(g')\,p\,\rho(g')^{-1} = \rho(h)\,p^{av}.$$

Now, $\rho(g)(V'') = \rho(g)((\mathrm{Id}_V - p^{av})(V)) = (\rho(g) - \rho(g)p^{av})(V) =$
$= (\rho(g) - p^{av}\rho(g))(V) = (\mathrm{Id} - p^{av})(\rho(g)(V)) \in (\mathrm{Id} - p^{av})(V) = V''$,
as announced $\boxtimes$

<u>Rmk</u>: This proof works for reps over any field k, provided that Char k does not divide $\# G$.

<u>Crl 2</u>: For finite G, $\forall\, V \in \mathrm{Rep}(G)$ is completely reducible (= semi-simple): $\boxed{V \cong V_1 \oplus \ldots \oplus V_s}$ for some $s \in \mathbb{N}$ and some $V_i \in \mathrm{Irrep}(G)$.

$\square$ Induction on $\dim_{\mathbb{C}}(V)$:
- $\dim_{\mathbb{C}}(V) = 1 \Rightarrow V \in \mathrm{Irrep}(G)$
- $\dim_{\mathbb{C}}(V) > 1 \Rightarrow$ either $V \in \mathrm{Irrep}(G)$, or $\exists$ a sub-rep. $V' \subsetneq V$, $V' \neq \{0\}$.
  In the 2nd case, Mashke $\Rightarrow V \cong V' \oplus V''$ as G-reps, with $\dim_{\mathbb{C}}(V') < \dim_{\mathbb{C}}(V)$, $\dim_{\mathbb{C}}(V'') < \dim_{\mathbb{C}}(V)$. $\boxtimes$

<u>Def.</u>: $V \in \mathrm{Rep}(G)$ is indecomposable if it has precisely two direct summands: V and $\{0\}$.

<u>Lemma 3</u>: Indecomposable $\Leftarrow$ irreducible.

$\square$ $V \cong V' \oplus V'' \Rightarrow V'$ is a sub-rep. of V $\boxtimes$

In general, $\not\Rightarrow$ (see Tutorial 1 & Homework 1).

In the proof of Crl 2, we actually showed:

<u>Crl 4</u>: $\boxed{\text{For finite } G, \text{ indecomposable} \Leftrightarrow \text{irreducible}}$