Arithmetic operations on functions

We’ll now learn how to construct complicated functions out of simpler ones.

First, two functions $f$ and $g$ can be added, subtracted, multiplied, and divided in a natural way.

✓ For $f + g$, $f - g$, and $fg$ to be defined, both $f$ and $g$ should be defined.

✓ For $f/g$ to be defined, both $f$ and $g$ should be defined, and also the value of $g$ should be non-zero: $g(x) \neq 0$. 
Example 1. Let \( f(x) = 1 + \sqrt{x - 2} \), and \( g(x) = x - 3 \). Then

\[
\begin{align*}
(f + g)(x) &= 1 + \sqrt{x - 2} + x - 3 = x - 2 + \sqrt{x - 2}, \\
(f - g)(x) &= 1 + \sqrt{x - 2} - (x - 3) = 4 - x + \sqrt{x - 2}, \\
(fg)(x) &= (1 + \sqrt{x - 2})(x - 3).
\end{align*}
\]

In all these cases, the domain is the intersection of the domains of \( f \) and \( g \), i.e., \([2, +\infty) \cap \mathbb{R} = [2, +\infty)\).

\[
(f/g)(x) = \frac{1 + \sqrt{x - 2}}{x - 3}.
\]

Here the domain is \([2, +\infty) \setminus \{3\} = [2, 3) \cup (3, +\infty)\).

In these examples, the domains of \( f + g \), \( f - g \), \( fg \), \( f/g \) are their natural domains. That is not always the case.
Example 2. Let \( f(x) = \sqrt{x - 2} \), and \( g(x) = \sqrt{x - 3} \). Then

\[
(fg)(x) = \sqrt{x - 2}\sqrt{x - 3} = \sqrt{(x - 2)(x - 3)} = \sqrt{x^2 - 5x + 6}.
\]

The domain of \( fg \) is \([2, +\infty) \cap [3, +\infty) = [3, +\infty)\).

The natural domain of \( \sqrt{x^2 - 5x + 6} \) is \((-\infty, 2] \cup [3, +\infty)\), since \( x^2 - 5x + 6 = (x - 2)(x - 3) \). This is different from \([3, +\infty)\)!

Example 3. Let \( f(x) = x \), and \( g(x) = 1/x \). Then

\[
(f/g)(x) = \frac{x}{1} = x^2.
\]

The domain of \( f/g \) is \((-\infty, 0) \cup (0, +\infty)\), which does not coincide with the natural domain of \( x^2 \), that is \((-\infty, +\infty)\).
Composition of functions

The arithmetic operations on functions were not “genuinely” new operations, since they just used arithmetics of real numbers at different points $x$ independently. Now we shall define a truly new way to construct new functions, not having numeric analogues.

**Definition.** The *composition* of two functions $f$ and $g$, denoted by $f \circ g$, is the function whose value at $x$ is $f(g(x))$:

$$(f \circ g)(x) = f(g(x)).$$

Its domain is defined as the set of all $x$ in the domain of $g$ for which the value $g(x)$ is in the domain of $f$.

Think about a cooking recipe: you apply the first step $g$ to your ingredients $x$, and then in the second step $f$ you can use only what you got at the first step, i.e., $g(x)$. The original ingredients $x$ are no longer available!
Composition of functions

Definition. The **composition** of two functions \( f \) and \( g \), denoted by \( f \circ g \), is the function whose value at \( x \) is \( f(g(x)) \):

\[
(f \circ g)(x) = f(g(x)).
\]

Its domain is defined as the set of all \( x \) in the domain of \( g \) for which the value \( g(x) \) is in the domain of \( f \).

**Example 1.** Recall the usual method for solving quadratic equations:

\[
x^2 + px + q = x^2 + 2\frac{p}{2}x + q = x^2 + 2\frac{p}{2}x + \frac{p^2}{4} - \frac{p^2}{4} + q = \left(x + \frac{p}{2}\right)^2 - \frac{p^2}{4} + q.
\]

It represents \( x^2 + px + q \) as the composition \( f(g(x)) \), where

\[
\begin{align*}
\checkmark \quad f(x) &= x^2 - \left(\frac{p^2}{4} - q\right), \\
\checkmark \quad g(x) &= x + \frac{p}{2}.
\end{align*}
\]
Example 2. Let \( f(x) = x^2 + 3 \) and \( g(x) = \sqrt{x} \). Then
\[
(f \circ g)(x) = (\sqrt{x})^2 + 3 = x + 3.
\]
Note that the domain of \( f \) is \((-\infty, +\infty)\), and the domain of \( g \) is \([0, +\infty)\), so the only restriction we impose on \( x \) to get the domain of \( f \circ g \) is that \( g \) is defined, and we conclude that \( (f \circ g)(x) = x + 3, x \geq 0 \).

⚠️ This function is different from \( h(x) = x + 3 \), whose natural domain is \( \mathbb{R} \)!

On the other hand,
\[
(g \circ f)(x) = \sqrt{x^2 + 3}.
\]
Since \( f \) is defined everywhere, the only restriction we impose on \( x \) to get the domain of \( g \circ f \) is that \( f(x) \) is in the domain of \( g \), so since \( x^2 + 3 \) is positive for all \( x \), we conclude that \( (g \circ f)(x) = \sqrt{x^2 + 3} \), with its natural domain \( \mathbb{R} = (-\infty, +\infty) \).

Question. What is the range of \( g \circ f \)?
Answer. \([\sqrt{3}, +\infty)\).
Let us plot some graphs to get a better feeling on how operations on functions work.

To begin with, we obtain the graph of $f(x) = \sqrt{x} + \frac{1}{x}$ from the graphs of $\sqrt{x}$ and $\frac{1}{x}$.
Translations and graph shifts

\[ y = x^2 \]
Let \( t \) denote the translation function, \( t(x) = x + a \). Replacing \( f(x) \) by \( f(x) + a = (t \circ f)(x) \) shifts the graph vertically: up if \( a > 0 \), down if \( a < 0 \).
Translations and graph shifts

\[ y = x^2 \]
Translations and graph shifts

Let $t$ denote the translation function, $t(x) = x + a$. Replacing $f(x)$ by $f(x + a) = (f \circ t)(x)$ shifts the graph horizontally: left if $a > 0$, right if $a < 0$. 

\[
\begin{align*}
y &= (x + 1)^2 \\
y &= x^2 \\
y &= (x - 1)^2
\end{align*}
\]
Let us plot the graph of the function $y = x^2 - 4x + 5$. By completing the square, we obtain $y = x^2 - 4x + 4 + 1 = (x - 2)^2 + 1$. So, our graph can be obtained from that of $y = x^2$ by a horizontal and a vertical shifts.
Reflections and graph symmetries

\[ y = \sqrt{x} \]
Let $r$ denote the reflection function, $r(x) = -x$. Replacing a function $f(x)$ by $f(-x) = (f \circ r)(x)$ reflects the graph about the $y$-axis, and replacing $f(x)$ by $-f(x) = (r \circ f)(x)$ reflects the graph about the $x$-axis.
Reflections and graph symmetries

Let us transform the graph of the function \( y = |x| \) into that of \( y = 4 - |x - 2| \).
By now, you have probably figured out the general principle: Composing a general function with a basic function corresponds to an elementary transformation of its graph.

<table>
<thead>
<tr>
<th>function transformation</th>
<th>graph transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) \rightarrow f(x + a)$</td>
<td>horizontal shift</td>
</tr>
<tr>
<td>$f(x) \rightarrow f(x) + a$</td>
<td>vertical shift</td>
</tr>
<tr>
<td>$f(x) \rightarrow f(-x)$</td>
<td>horizontal reflection (about the $y$-axis)</td>
</tr>
<tr>
<td>$f(x) \rightarrow -f(x)$</td>
<td>vertical reflection (about the $x$-axis)</td>
</tr>
<tr>
<td>$f(x) \rightarrow f(cx)$</td>
<td>horizontal compression/stretching</td>
</tr>
<tr>
<td>$f(x) \rightarrow cf(x)$</td>
<td>vertical compression/stretching</td>
</tr>
</tbody>
</table>

$c > 1$: vertical stretching, horizontal compression;
$0 < c < 1$: vertical compression, horizontal stretching.
Scaling and graph compression/stretching

\[ y = \cos x \]
Let $s$ denote the scaling function, $s(x) = cx$. Replacing a function $f(x)$ by $cf(x) = (s \circ f)(x)$ stretches the graph vertically if $c > 1$, and compresses the graph vertically if $0 < c < 1$. 
Scaling and graph compression/stretching

\[ y = \cos(x) \]
Let $s$ denote the scaling function, $s(x) = cx$. Replacing a function $f(x)$ by $f(cx) = (f \circ s)(x)$ compresses the graph horizontally if $c > 1$, and stretches the graph horizontally if $0 < c < 1$. 

$\frac{\frac{\frac{\frac{c}{c}}{c}}{c}}{c} \frac{\frac{\frac{c}{c}}{c}}{c}$