Lecture 3: New functions from old ones

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MA1S11A: Calculus with Applications for Scientists

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1 Arithmetic operations on functions

We'll now learn how to construct complicated functions out of simpler ones.

First, two functions f and g can be added, subtracted, multiplied, and divided in a natural way.

✓ For $f + g$, $f - g$, and fg to be defined, both f and g should be defined.

✓ For f/g to be defined, both f and g should be defined, and also the value of g should be non-zero: $g(x) \neq 0$.

1 Arithmetic operations on functions

Example 1. Let $f(x) = 1 + \sqrt{x-2}$, and $g(x) = x - 3$. Then

- ✓ $(f+g)(x) = 1 + \sqrt{x-2} + x - 3 = x - 2 + \sqrt{x-2}$,
- ✓ $(f-g)(x) = 1 + \sqrt{x-2} - (x-3) = 4 - x + \sqrt{x-2}$,
- ✓ $(fg)(x) = (1 + \sqrt{x-2})(x-3)$.

In all these cases, the domain is the intersection of the domains of f and g, i.e., $[2, +\infty) \cap \mathbb{R} = [2, +\infty)$.

- ✓ $(f/g)(x) = \frac{1+\sqrt{x-2}}{x-3}$.

Here the domain is $[2, +\infty) \setminus \{3\} = [2, 3) \cup (3, +\infty)$.

In these examples, the domains of $f + g$, $f - g$, fg, f/g are their natural domains. That is not always the case.

Example 2. Let $f(x) = \sqrt{x-2}$, and $g(x) = \sqrt{x-3}$. Then

$$(fg)(x) = \sqrt{x-2}\sqrt{x-3} = \sqrt{(x-2)(x-3)} = \sqrt{x^2-5x+6}.$$

The domain of fg is $[2, +\infty) \cap [3, +\infty) = [3, +\infty)$.

The natural domain of $\sqrt{x^2-5x+6}$ is $(-\infty, 2] \cup [3, +\infty)$, since $x^2 - 5x + 6 = (x-2)(x-3)$. This is different from $[3, +\infty)$!

Example 3. Let $f(x) = x$, and $g(x) = 1/x$. Then

$$(f/g)(x) = \frac{x}{\frac{1}{x}} = x^2.$$

The domain of f/g is $(-\infty, 0) \cup (0, +\infty)$, which *does not* coincide with the natural domain of x^2, that is $(-\infty, +\infty)$.

2 Composition of functions

The arithmetic operations on functions were not “genuinely” new operations, since they just used arithmetics of real numbers at different points x independently. Now we shall define a truly new way to construct new functions, not having numeric analogues.

Definition. The **composition** of two functions f and g, denoted by $f \circ g$, is the function whose value at x is $f(g(x))$:

$$(f \circ g)(x) = f(g(x)).$$

Its domain is defined as the set of all x in the domain of g for which the value $g(x)$ is in the domain of f.

Think about a cooking recipe: you apply the first step g to your ingredients x, and then in the second step f you can use only what you got at the first step, i.e., $g(x)$. The original ingredients x are no longer available!

2 Composition of functions

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Example 1. Recall the usual method for solving quadratic equations:

$$x^2+px+q = x^2+2\frac{p}{2}x+q = x^2+2\frac{p}{2}x+\frac{p^2}{4}-\frac{p^2}{4}+q = \left(x+\frac{p}{2}\right)^2-\frac{p^2}{4}+q.$$

It represents x^2+px+q as the composition $f(g(x))$, where

- ✓ $f(x) = x^2 - \left(\frac{p^2}{4} - q\right)$,
- ✓ $g(x) = x + \frac{p}{2}$.

2 Composition of functions

Example 2. Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Then

$$(f \circ g)(x) = (\sqrt{x})^2 + 3 = x + 3.$$

Note that the domain of f is $(-\infty, +\infty)$, and the domain of g is $[0, +\infty)$, so the only restriction we impose on x to get the domain of $f \circ g$ is that g is defined, and we conclude that $(f \circ g)(x) = x + 3, x \geqslant 0$.

⚠ This function is different from $h(x) = x + 3$, whose natural domain is $\mathbb{R}$!

On the other hand,

$$(g \circ f)(x) = \sqrt{x^2 + 3}.$$

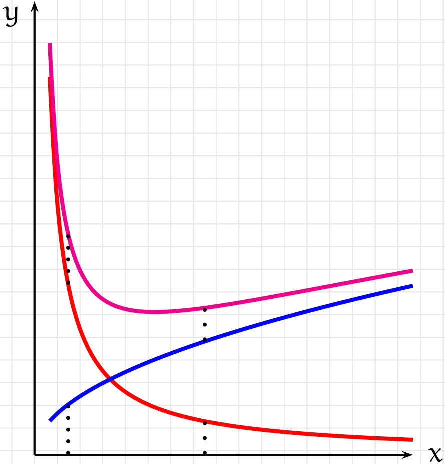
Since f is defined everywhere, the only restriction we impose on x to get the domain of $g \circ f$ is that $f(x)$ is in the domain of g, so since $x^2 + 3$ is positive for all x, we conclude that $(g \circ f)(x) = \sqrt{x^2 + 3}$, with its natural domain $\mathbb{R} = (-\infty, +\infty)$.

Question. What is the range of $g \circ f$?
Answer. $[\sqrt{3}, +\infty)$.

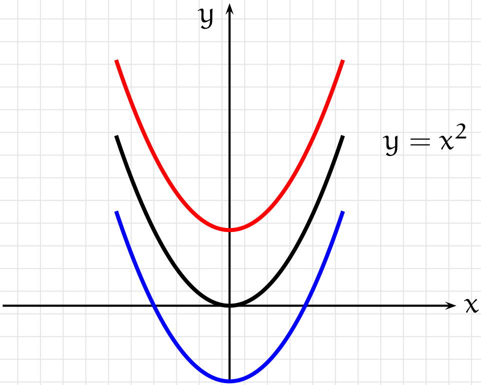
3 Operations on functions and graphs

Let us plot some graphs to get a better feeling on how operations on functions work.

To begin with, we obtain the graph of $f(x) = \sqrt{x} + \frac{1}{x}$ from the graphs of $\sqrt{x}$ and $\frac{1}{x}$.



4 Translations and graph shifts



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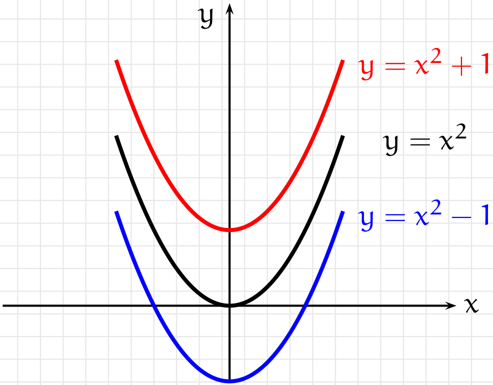
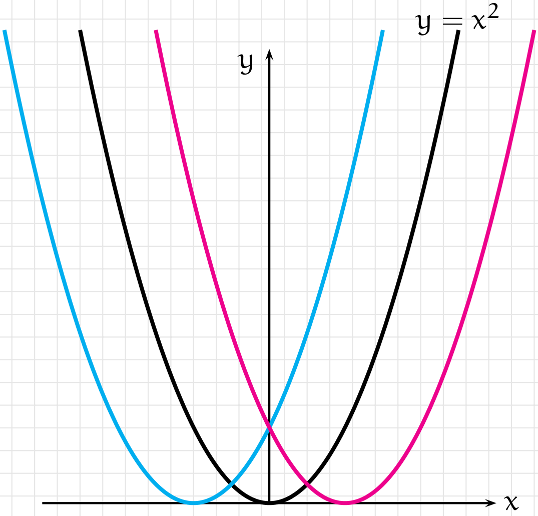


Let t denote the translation function, $t(x) = x + a$. Replacing $f(x)$ by $f(x) + a = (t \circ f)(x)$ shifts the graph vertically: up if $a > 0$, down if $a < 0$.

4 Translations and graph shifts



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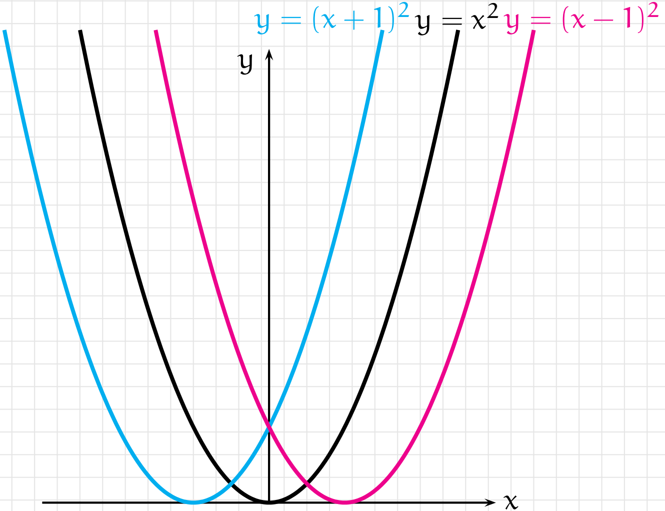
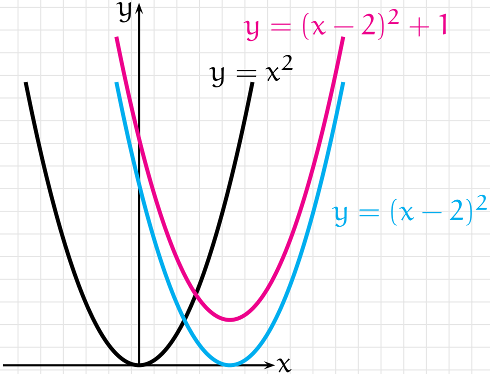


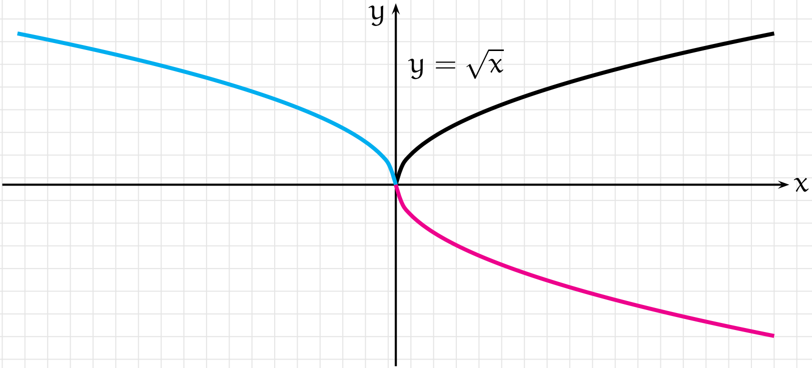
Let t denote the translation function, $t(x) = x + a$. Replacing $f(x)$ by $f(x+a) = (f \circ t)(x)$ shifts the graph horizontally: left if $a > 0$, right if $a < 0$.

4 Translations and graph shifts

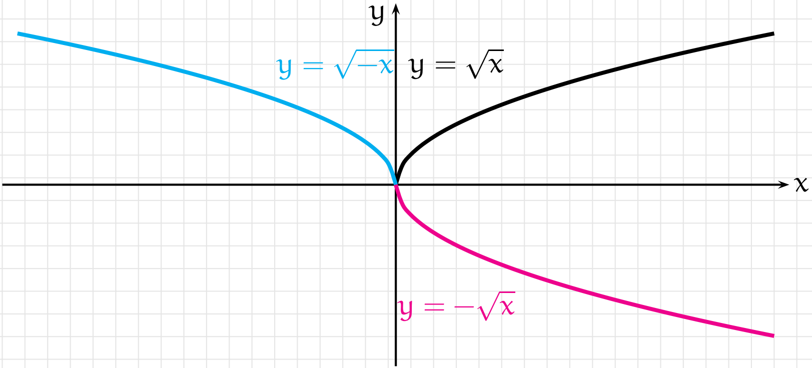
Let us plot the graph of the function $y = x^2 - 4x + 5$. By completing the square, we obtain $y = x^2 - 4x + 4 + 1 = (x - 2)^2 + 1$. So, our graph can be obtained from that of $y = x^2$ by a horizontal and a vertical shifts.



5 Reflections and graph symmetries



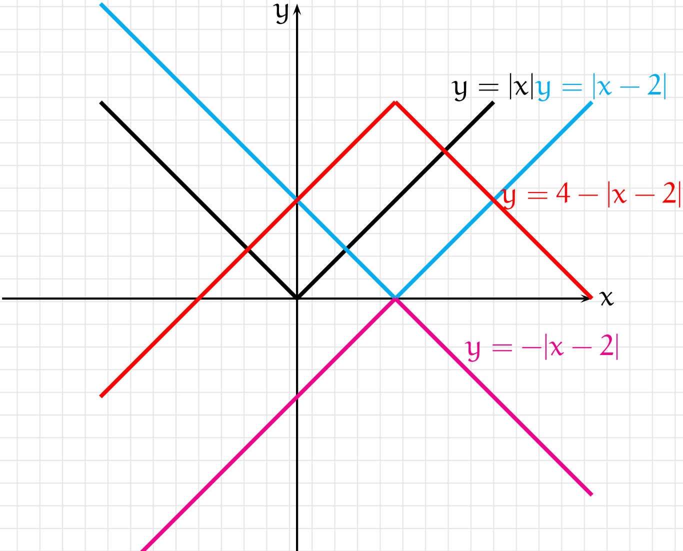
5 Reflections and graph symmetries



Let r denote the reflection function, $r(x) = -x$. Replacing a function $f(x)$ by $f(-x) = (f \circ r)(x)$ reflects the graph about the y-axis, and replacing $f(x)$ by $-f(x) = (r \circ f)(x)$ reflects the graph about the x-axis.

5 Reflections and graph symmetries

Let us transform the graph of the function $y = |x|$ into that of $y = 4 - |x - 2|$.



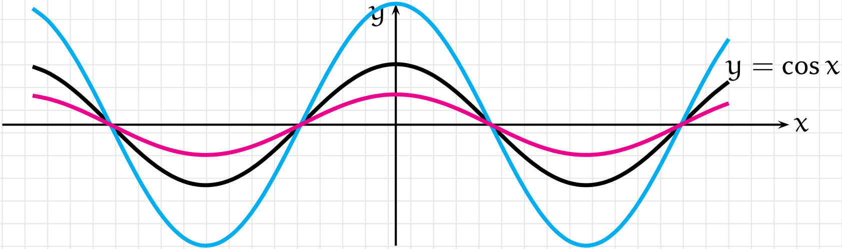
6 Transforming functions and graphs

By now, you have probably figured out the general principle:
Composing a general function with a basic function corresponds to an elementary transformation of its graph.

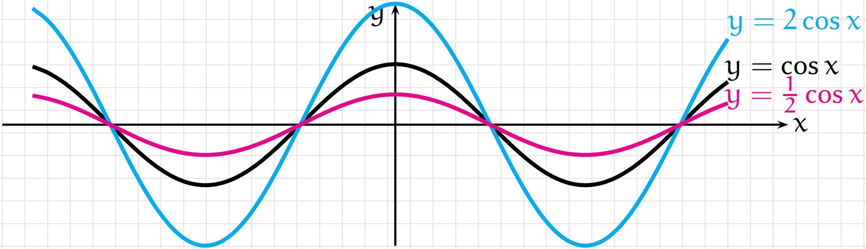
function transformation	graph transformation
$f(x) \rightsquigarrow f(x+a)$	horizontal shift
$f(x) \rightsquigarrow f(x)+a$	vertical shift
$f(x) \rightsquigarrow f(-x)$	horizontal reflection (about the y-axis)
$f(x) \rightsquigarrow -f(x)$	vertical reflection (about the x-axis)
$f(x) \rightsquigarrow f(cx)$	horizontal compression/stretching
$f(x) \rightsquigarrow cf(x)$	vertical compression/stretching

$c > 1$: vertical stretching, horizontal compression;
$0 < c < 1$: vertical compression, horizontal stretching.

7 Scaling and graph compression/stretching

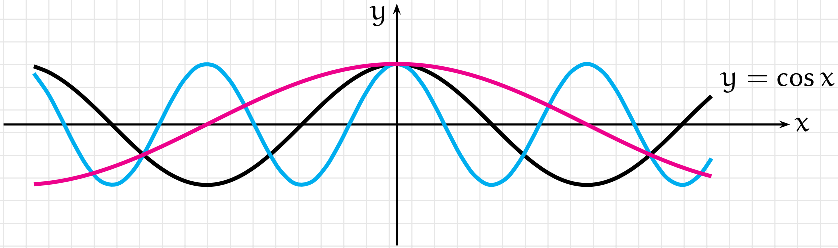


7 Scaling and graph compression/stretching



Let s denote the scaling function, $s(x) = cx$. Replacing a function $f(x)$ by $cf(x) = (s \circ f)(x)$ stretches the graph vertically if $c > 1$, and compresses the graph vertically if $0 < c < 1$.

7 Scaling and graph compression/stretching



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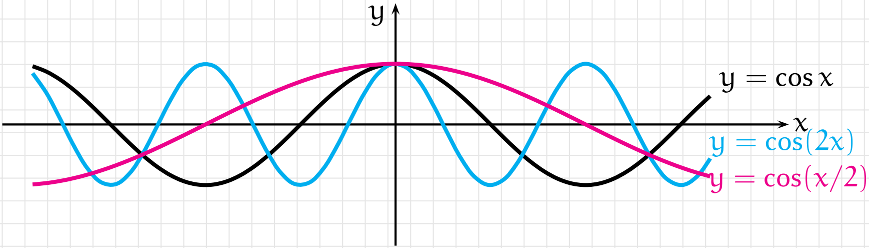

Let s denote the scaling function, $s(x) = cx$. Replacing a function $f(x)$ by $f(cx) = (f \circ s)(x)$ compresses the graph horizontally if $c > 1$, and stretches the graph horizontally if $0 < c < 1$.