Lecture 28:
Applications of integral calculus in sciences

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MA1S11A: Calculus with Applications for Scientists

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1 Derivatives and rectilinear motion

One of the original motivations for introducing derivatives was to handle velocity and acceleration. To cover these concepts in full generality, we need to work with vector-valued functions. But real-valued functions also apply in practice—to study **rectilinear motion**, i.e., motion along a line.

Consider for instance the height of an object thrown vertically, either upward or downward. From physics, you know that this motion is parabolic, and can be described by the following functions:

$$\text{position (height) } x(t) = at^2 + bt + c,$$
$$\text{velocity } v(t) = x'(t) = 2at + b,$$
$$\text{acceleration } a(t) = x''(t) = 2a,$$

where $a, b, c \in \mathbb{R}$ are parameters. Here t is the time variable.

From physics you know that in this situation the acceleration is due to gravity only, and near the surface of the Earth its value is $\approx 9.8\,\mathrm{m}/\mathrm{s}^2$. So, you can take $a = -4.9\,\mathrm{m}/\mathrm{s}^2$. (The sign is negative since gravity acceleration is directed downward).

1 Derivatives and rectilinear motion

The motion equation becomes

$$x(t) = -4.9t^2 + bt + c.$$

We can measure the height from the point where the object was thrown, and start measuring time when this happened. This means $c = x(0) = 0$. The equation of motion now becomes

$$x(t) = -4.9t^2 + bt.$$

To determine b, you need some extra information. For example, if you are given the initial velocity v_0, then

$$b = x'(0) = v_0.$$

Let us find stationary points of $x(t)$. Equation $x'(t) = 0$ means here $-9.8t + b = 0$, that is, $t = \frac{b}{9.8}$. If the original movement direction was upward, we get $b = v_0 > 0$, so $t = \frac{b}{9.8} > 0$ is a valid value for time. Looking at the sign of $x'(t) = -9.8t + b$, we see that in this case $x(t)$ has a local maximum at $\frac{b}{9.8}$. This corresponds to what you know from practice: a ball thrown upward stops at some moment and then falls down.

Exercise. Plot the graph of $h(t)$ for $v_0 = 5\,m/s^2$ and $v_0 = -5\,m/s^2$.

2 Integrals and rectilinear motion

Now, integrals appear in different situations involving rectilinear motion.

1) Suppose that you are driving a car looking at your speedometer. From its readings only, you would like to know

(a) how far you got,

(b) how may kilometres you have driven,

in both cases between the beginning of your trip ($t = 0$) and its end ($t = t_1$).

The answers are given by integrals:

(a) the displacement $\int_0^{t_1} v(t)\, dt = x(t_1) - x(0)$;

(b) the distance traveled $\int_0^{t_1} |v(t)|\, dt$.

2 Integrals and rectilinear motion

2) Now, suppose that you know the acceleration of the object. For instance, let it be constant, $a(t) = c_0$ (as in the free fall considered above). From this you deduce

$$v(t) = \int a(t)\,dt = c_0 t + c_1,$$

$$x(t) = \int v(t)\,dt = \frac{c_0}{2}t^2 + c_1 t + c_2$$

for some real constants c_1, c_2. We thus recover the parabolic motion known from physics.

The constants can be computed using some additional data, such as

- ✓ the initial position $x(0) = c_2$,
- ✓ the initial velocity $v(0) = c_1$.

3) Finally, to find the average position (e.g., height) of the object during its movement, you also need integrals:

$$x_{ave} = \frac{1}{t_1}\int_0^{t_1} x(t)\,dt.$$

3 Integrals and work

Basic principle: If a constant force of magnitude F, applied in the direction of the motion of an object, moves this object by a distance d, then the **work** performed by the force on the object is defined to be

$$W = F \cdot d.$$

Using that principle and our usual divide-and-approximate technique, one can define and compute the work of a variable force:

1) divide the path of the object into many small parts;

2) on each part, assume the force to be constant and apply the above principle;

3) add the results together;

4) compute the limit of the total sums when the parts get arbitrarily small.

3 Integrals and work

As a consequence, the work is equal to limit of Riemann sums:

$$\sum_{i=1}^{N} F(x_i^*)\Delta x_i,$$

that is,

$$W = \int_a^b F(x)\,dx,$$

where a and b are the initial and the final position of the object, respectively.

4 Example: Hooke's Law

Hooke's Law: a spring stretched x units beyond its natural length pulls back with a force

$$F(x) = kx,$$

where k is a constant called the stiffness of the spring; it depends on the material as well as the thickness of the spring.

Example. Suppose that a spring exerts a force of 5 N when stretched one metre beyond its natural length. Find the work required to stretch the spring 1.8 metres beyond its natural length.

Let us first compute the stiffness of this spring.
Applying Hooke's Law with $F = 5$, $x = 1$, we get $k = 5$ (N/m).

Now, the work required is

$$W = \int_a^b F(x)\,dx = \int_0^{1.8} 5x\,dx = \frac{5x^2}{2}\Bigg]_0^{1.8} = 8.1\ (\mathrm{N}\cdot\mathrm{m}).$$

5 Work and energy

Let us assume that an object of mass m moves along the x axis as a result of the force $F(t)$ applied in the direction of motion.

The motion of the object is described by

- ✓ its position $x(t)$ at the time t;
- ✓ its instantaneous velocity $v(t) = x'(t)$;
- ✓ its instantaneous acceleration $a(t) = v'(t)$.

Newton's Second Law of Motion: If an object of mass m is moving as a result of a force F applied to it, then that object undergoes an acceleration

$$a(t) = \frac{F(t)}{m}.$$

This means that the mass m, the force F, and some initial data (e.g. the position $x(0)$ and the velocity $v(0)$ at $t = 0$) completely determine the movement of the object.

5 Work and energy

Now, suppose that

- ✓ at the initial moment t_0 the object is at the position $x(t_0) = x_0$ moving with the initial velocity $v(x_0) = v_0$;
- ✓ at the final moment t_1, we have $x(t_1) = x_1$ and $v(x_1) = v_1$.

The work of the force moving the object is $\int\limits_{x_0}^{x_1} F(x)\,dx$. Recalling that the position x is a function of the time t, we get

$$W = \int_{x(t_0)}^{x(t_1)} F(x)\,dx = \int_{t_0}^{t_1} F(x(t))x'(t)\,dt = \int_{t_0}^{t_1} ma(t)v(t)\,dt$$
$$= \int_{t_0}^{t_1} mv'(t)v(t)\,dt = \int_{v(t_0)}^{v(t_1)} mv\,dv = \frac{mv^2}{2}\Bigg]_{v_0}^{v_1} = \frac{mv_1^2}{2} - \frac{mv_0^2}{2}.$$

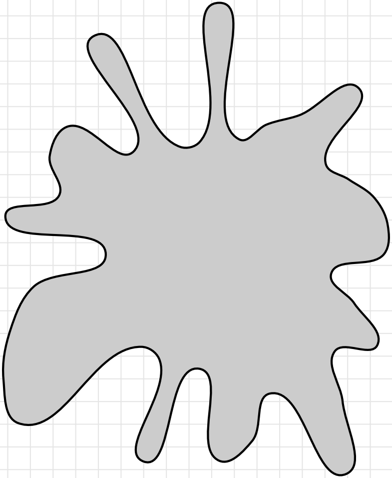
The quantity $\frac{mv^2}{2}$ is called the **kinetic energy** of an object.

We just established a famous law from physics:

the work performed by the force on the object is equal to the change in the kinetic energy of the object.

6 Integrals and centre of gravity

Consider a lamina (a flat object thin enough to be viewed as a 2d plane region), which we suppose homogeneous (composed uniformly throughout)

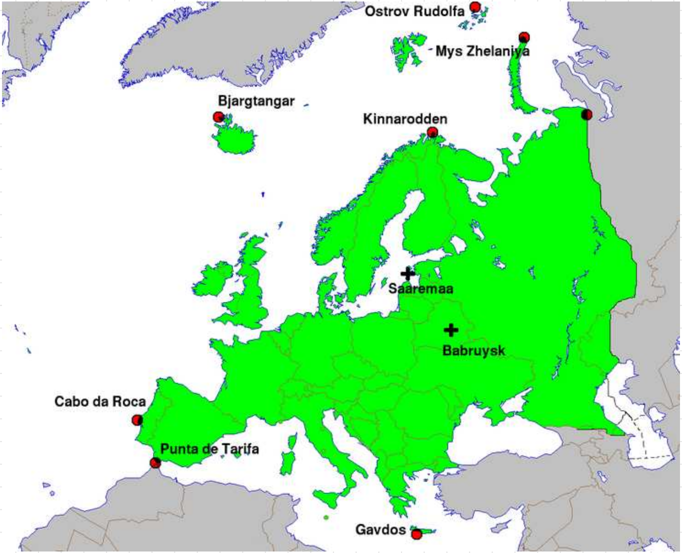


6 Integrals and centre of gravity

It can be shown that each lamina has a **centre of gravity**, that is, a point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is equivalent to that of a single force acting at the point $(\bar{x}, \bar{y})$.

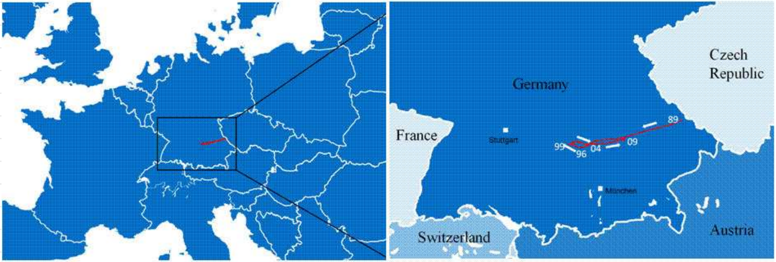
For a symmetric lamina, like a circle, or a square, the centre of gravity coincides with the symmetry centre, but for a more complex shape it is not as obvious.

For instance, what is the centre of gravity of Europe?
Many places claim this title. According to the Guinness World Records, the winner is located in... Lithuania!

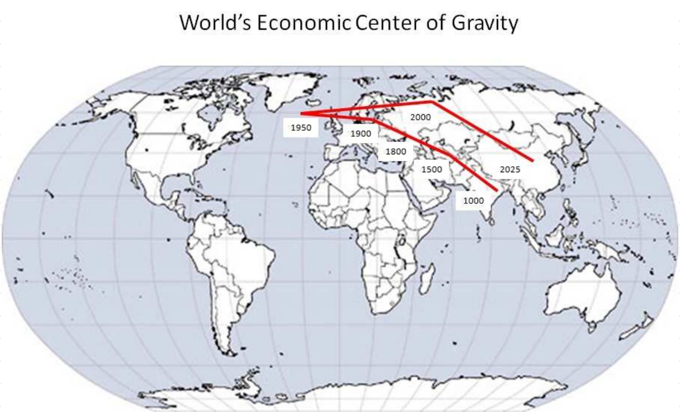


Another centre of gravity:

Path of European Economic Centre of Gravity (ECG) 1989 - 2009



And one more:



World's Economic Center of Gravity
1950
1900
1800
1500
1000
2000
2025

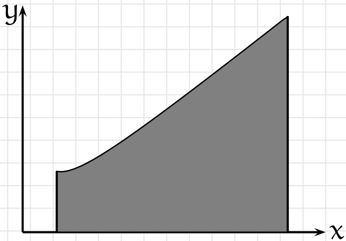
6 Integrals and centre of gravity

From basic mechanics, one shows that for a lamina whose mass is localised at finitely many points $A_1, \dots, A_n$ (with masses $m_1, \dots, m_n$), its centre of gravity M is determined from the equilibrium conditions

$$m_1(\overrightarrow{OA_1} - \overrightarrow{OM}) + m_2(\overrightarrow{OA_2} - \overrightarrow{OM}) + \cdots + m_n(\overrightarrow{OA_n} - \overrightarrow{OM}) = 0.$$

For a general lamina, we have to "sum over infinitely many points". As usual, this is done by integrating.

Let us first compute the centre of gravity of a lamina occupying the region bounded by a graph $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.



6 Integrals and centre of gravity

Let us divide the interval $[a, b]$ into many small parts, approximating the lamina by a union of rectangles.

For each small rectangle,

✓ its centre of gravity is at the point $(x_k^*, \frac{1}{2}f(x_k^*))$, where x_k^* is the midpoint of its base;

✓ its mass is $\Delta m_k = \delta f(x_k^*)\Delta x_k$, where δ is the density of our lamina (i.e., its mass per unit area).

So, the gravity centre equilibrium conditions are

$$\sum_{k=1}^{n}(x_k^* - \bar{x})\delta f(x_k^*)\Delta x_k = 0, \qquad \sum_{k=1}^{n}\left(\frac{1}{2}f(x_k^*) - \bar{y}\right)\delta f(x_k^*)\Delta x_k = 0.$$

As the mesh size of the partition of $[a, b]$ gets smaller, the equations take the limit form

$$\int_a^b (x - \bar{x})\delta f(x)\,dx = 0, \qquad \int_a^b \left(\frac{1}{2}f(x) - \bar{y}\right)\delta f(x)\,dx = 0.$$

6 Integrals and centre of gravity

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Recalling that $\bar{x}$ and $\bar{y}$ are constants, these can be written as

$$\int_a^b \delta x f(x)\,dx = \bar{x}\int_a^b \delta f(x)\,dx, \qquad \int_a^b \frac{1}{2}\delta(f(x))^2\,dx = \bar{y}\int_a^b \delta f(x)\,dx.$$

Examining these formulas, we notice that:

- ✓ the constant factor $\delta > 0$ can be dropped;
- ✓ $\int\limits_a^b f(x)\,dx$ is the area of the lamina.

Finally, we get the formulas

$$\bar{x} = \frac{\int_a^b xf(x)\,dx}{\text{area of the lamina}}, \qquad \bar{y} = \frac{\frac{1}{2}\int_a^b f(x)^2\,dx}{\text{area of the lamina}}.$$

6 Integrals and centre of gravity

Example. Assume that the lamina is a half-circle $0 \leqslant y \leqslant \sqrt{1-x^2}$. We have

$$\bar{x} = \frac{\int_{-1}^{1} x\sqrt{1-x^2}\,dx}{\text{area of the lamina}}, \qquad \bar{y} = \frac{\frac{1}{2}\int_{-1}^{1} 1-x^2\,dx}{\text{area of the lamina}}.$$

✓ Since $\bar{x}$ is proportional to the integral of an odd function $x\sqrt{1-x^2}$, it is zero.

✓ For $\bar{y}$, we have

$$\bar{y} = \frac{\frac{1}{2}\int_{-1}^{1} 1-x^2\,dx}{\frac{1}{2}\pi} = \frac{\left. x-\frac{x^3}{3}\right]_{-1}^{1}}{\pi} = \frac{1-\frac{1}{3}+1-\frac{1}{3}}{\pi} = \frac{4}{3\pi}.$$

So, the centre of gravity of our half-circle is situated at the point $(0, \frac{4}{3\pi})$.

Exercise. Compute $\int_a^b x\sqrt{1-x^2}\,dx$.