Lecture 27:
Applications of integral calculus in maths

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MA1S11A: Calculus with Applications for Scientists

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1 Area between two curves

Our last big topic is applications of definite integrals in maths and sciences.

The main application, which motivated the whole development of integral calculus, is the computation of areas, and, in the same spirit, of volumes, surfaces, and curve lengths.

Recall that the area A below the graph of a continuous non-negative function f on an interval $[a, b]$ can be computed as the definite integral of f:

$$A = \int_a^b f(x)\,dx.$$

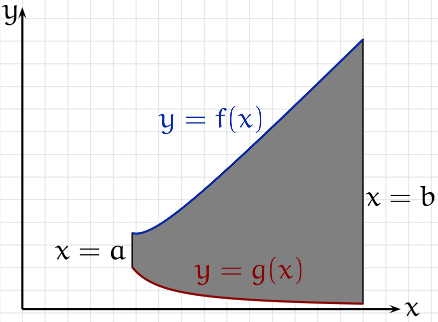
1 Area between two curves

The second simplest case is the area A of the region bounded by

- ✓ the graphs of two continuous functions f and g on $[a, b]$,
- ✓ and the lines $x = a$, $x = b$.

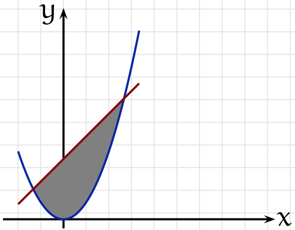
Here we assume $f(x) \geqslant g(x)$ on $[a, b]$. This area is also a definite integral:

$$A = \int_a^b f(x) - g(x)\,dx.$$



1 Area between two curves

Example. Find the area of the region enclosed between the parabola $y = x^2$ and the line $y = x + 2$.



To find the intersection points of the two graph, solve the equation $x^2 = x + 2$. Its solutions are $x = 2$ and $x = -1$, so the intersection points are $(2, 4)$ and $(-1, 1)$. The area in question is then computed by the integral

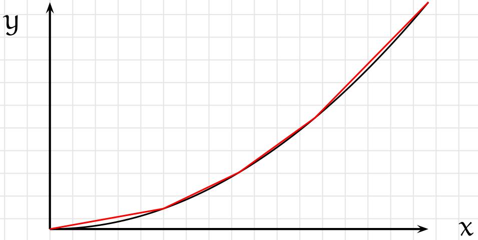
$$\int_{-1}^{2} (x+2) - x^2 \, dx = \left. \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5.$$

Exercise. Find the area of the region enclosed between the graphs of $y = xe^x$ and $y = x^2 e^x$.

2 Arc length

The arc length of a plane curve $y = f(x)$ is defined following the same scheme as the area of a plane figure:

1) approximate the curve by polygonal lines:



2) compute the lengths of these lines (using the Pythagorean Theorem):

$$\sum_{k=1}^{N} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{N} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k;$$

3) find the limit of these approximations, if it exists:

$$\lim_{N \to +\infty} \sum_{k=1}^{N} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

2 Arc length

Theorem. The arc length of the curve $y = f(x)$ on $[a, b]$ is equal to

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

provided $f'(x)$ is continuous on $[a, b]$.

Example 1. Compute the arc length of the curve $y = x^{3/2}$ on $[1, 2]$.

Since $(x^{3/2})' = \frac{3}{2}\sqrt{x}$, the desired arc length is $\displaystyle\int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx$.

Using the substitution

$$u = 1 + \frac{9}{4}x, \quad du = \frac{9}{4}\, dx, \quad u(1) = \frac{13}{4}, \quad u(2) = \frac{22}{4},$$

we get

$$\int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{4}{9}\int_{13/4}^{22/4} u^{1/2} du = \frac{8}{27}u^{3/2}\Big]_{13/4}^{22/4} = \frac{22\sqrt{22} - 13\sqrt{13}}{27}$$

$$\approx 2.1.$$

2 Arc length

Example 2. Compute the arc length of the half-circle $x^2 + y^2 = 1$, $y \geqslant 0$.

For $y \geqslant 0$, the relation $x^2 + y^2 = 1$ is equivalent to $y = \sqrt{1 - x^2}$, which describes a curve on $[-1, 1]$.

Putting $f(x) = \sqrt{1 - x^2}$, we get $f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}$. This derivative has infinite discontinuities at $x = \pm 1$.

Take any a and b satisfying $-1 < a < b < 1$. We have

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \frac{x^2}{1 - x^2}}\, dx = \int_a^b \frac{dx}{\sqrt{1 - x^2}} = [\arcsin x]_a^b.$$

Recall that the function arcsin is continuous. So, when a approaches -1 and b approaches 1, the above length approaches

$$[\arcsin x]_{-1}^{1} = \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi.$$

We get a rigorous proof of the fact you know from school:

the length of the unit circle is 2π.

3 Computing sums

Recall that definite integrals are

- ✓ defined as the limits of Riemann sums;
- ✓ mainly computed using indefinite integrals.

Thus, one can compute certain sums by presenting them as Riemann sums and finding the appropriate indefinite integral.

Example. The harmonic series

$$\sum_{k=1}^{+\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is omnipresent in maths. It is therefore important to have good estimates for its partial sums $\sum_{k=1}^{n} \frac{1}{k}$.

3 Computing sums

These two figures prove that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > \int_1^{n+1}\frac{dx}{x} > \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1}.$$

Let us compute the integral in the middle:

$$\int_1^{n+1}\frac{dx}{x} = [\ln(x)]_1^{n+1} = \ln(n+1)-\ln(1) = \ln(n+1),$$

so we have

$$\ln(n+1) < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < \ln(n+1)+1-\frac{1}{n+1} < \ln(n+1)+1.$$

In particular, we see that the partial harmonic sums grow slowly (logarithmic growth), but become as large as we want as n increases.

4 Average value of a function

Recall that the (arithmetic) average of N numbers $a_1, a_2, \ldots, a_N$ is

$$\frac{1}{N}\sum_{k=1}^{N} a_N = \frac{1}{N}(a_1 + a_2 + \cdots + a_N).$$

But how to define the average f_{ave} of a function f on $[a, b]$? This involves "summing infinitely many numbers", which can be done by integrating!

Concretely, reasonable approximations to f_{ave} are the averages of the values of f at points $x_1^*, \ldots, x_N^*$ equally spaced on $[a, b]$:

$$\frac{1}{N}\sum_{k=1}^{N} f(x_k^*).$$

These expressions should remind you of the Riemann sums

$$\sum_{k=1}^{N} f(x_k^*)\frac{b-a}{N} = (b-a)\cdot\frac{1}{N}\sum_{k=1}^{N} f(x_k^*),$$

which for large N approximate the integral $\int_a^b f(x)\,dx$, if it exists.

4 Average value of a function

This motivates the following

Definition. The **average value** of a continuous function f on $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

By the Mean Value Theorem for Integrals, $f_{ave} = f(c)$ for some point $c \in [a, b]$.

Example 1. If the functions $x(t)$ and $v(t) = x'(t)$ describe the position and the velocity of an object moving along a line, then the average value of the velocity function on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b v(t)\,dt = \frac{x(b)-x(a)}{b-a},$$

which is the average velocity of the object between $t = a$ and $t = b$.

4 Average value of a function

Example 2. The average value of a linear function $f(x) = mx + k$ on $[a, b]$ is

$$\begin{aligned}
\frac{1}{b-a}\int_a^b mx + k\,dx &= \frac{1}{b-a}\left[m\frac{x^2}{2} + kx\right]_a^b \\
&= \frac{1}{b-a}\left(m\frac{b^2-a^2}{2} + k(b-a)\right) \\
&= m\frac{b+a}{2} + k = f(\frac{a+b}{2}),
\end{aligned}$$

which is the value f takes in the middle of the interval $[a, b]$.

4 Average value of a function

Example 3. The average value of a the function $f(x) = \sin(x)$ on $[0, \frac{\pi}{2}]$ is

$$\frac{1}{\pi/2}\int_0^{\frac{\pi}{2}} \sin(x)\,dx = \frac{2}{\pi}\left[-\cos(x)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}(-\cos(\frac{\pi}{2}) + \cos(0)) = \frac{2}{\pi}.$$

Observe that here the average value is

- ✓ neither the midpoint $\frac{1}{2}$ of the range $[0, 1]$ of the function $\sin(x)$ on $[0, \frac{\pi}{2}]$,
- ✓ nor the value $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ taken in the middle of the interval $[0, \frac{\pi}{2}]$.

The number we computed can be interpreted as follows: if you randomly choose sufficiently many points in the first quadrant part of the unit circle, then the average value of their y-coordinates will be close to $\frac{2}{\pi}$.

More generally, integrals are essential when computing probabilities.