26-27. Representation theory of general linear groups

In the last lectures, we will look at another family of infinite groups — the **general linear groups**

$GL_n := GL_n(\mathbb{C}) = Mat^*_{n\times n}(\mathbb{C}) \cong Aut_{\mathbb{C}}(\mathbb{C}^n)$.

The main brick of their representation theory is the natural rep.

$V = (\mathbb{C}^n, Id)$, where Id is the identity map $GL_n \to GL_n$.

It is faithful, since Id is an injective map.

Recall that for a finite group G and a faithful $V \in Rep(G)$, one can recover the whole $Irrep(G)$ by decomposing into irreps the tensor powers $V^{\otimes n}$. For instance, when studying $Rep(S_n)$, we extensively used its faithful standard rep. V^{st}.

For the GL_n the situation turns out to be similar, even though these groups are infinite.

Thus, $\forall k \in \mathbb{N}$, $V^{\otimes k} \in Rep(GL_n)$, with $g\cdot(v_1 \otimes \dots \otimes v_k) = g\cdot v_1 \otimes \dots \otimes g\cdot v_k$.

These tensor powers $V^{\otimes k}$ also carry another action:

Lemma 38:
- $V^{\otimes k} \in Rep(S_k)$, with $\sigma\cdot(v_1 \otimes \dots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(k)}$
- $\forall g \in GL_n$, $\forall \sigma \in S_k$, $\forall \bar{v} \in V^{\otimes n}$, $g\cdot(\sigma\cdot\bar{v}) = \sigma\cdot(g\cdot\bar{v})$;
  that is, the GL_n- and the S_k-actions on $V^{\otimes k}$ commute.

The proof is straightforward.

Note that the definition of the S_k-action will not work if the subscripts $\sigma^{-1}(i)$ are replaced with $\sigma(i)$.

In fact, the connection between the GL_n- and the S_k-actions on $V^{\otimes k}$ is extremely strong: they are mutual centralisers. That is,

$$\boxed{End_{S_k}(V^{\otimes k}) \cong \mathbb{C}GL_n \qquad \& \qquad End_{GL_n}(V^{\otimes k}) \cong \mathbb{C}S_k}.$$

Here we used the following notations:

- for a group G, its group algebra is the vector space $\mathbb{C}G = \bigoplus_{g\in G} \mathbb{C}e_g$, with the algebra operations
  - $(\sum_g \alpha_g e_g) + (\sum_g \beta_g e_g) = \sum_g (\alpha_g + \beta_g) e_g$,
  - $(\sum_g \alpha_g e_g)\cdot(\sum_h \beta_h e_h) = \sum_k (\sum_{\substack{g,h \text{ s.t.} \\ gh=k}} \alpha_g \beta_h) e_k$;
- for $(V,\rho) \in Rep(G)$, we abusively denote by $\mathbb{C}G$ the linear

span of $\rho(G) \subseteq \mathrm{Aut}_{\mathbb{C}}(V)$ in $\mathrm{End}_{\mathbb{C}}(V)$; it can also be seen as $\hat{\rho}(\mathbb{C}G)$, where $\hat{\rho}: \mathbb{C}G \to \mathrm{End}_{\mathbb{C}}(V)$ is the obvious extension of the group action ρ into an algebra action.

From this connection and our complete understanding of $\mathrm{Rep}(S_k)$, one can deduce crucial information on $\mathrm{Rep}(GL_n)$:

Schur-Weyl duality: $\forall\, k, n \in \mathbb{N}$,

$$\boxed{V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k, \\ p(\lambda) \le n}} V_\lambda \otimes W_{n;\lambda}}$$ both as S_k- and GL_n-reps,

where
- $p(\lambda) = \#\{\text{parts of } \lambda\} = \#\{\text{rows in the Young diag. of } \lambda\}$
- the V_λ are the Specht reps of S_k
- the $W_{n;\lambda}$ are the Weyl reps of GL_n (they are also called Weyl modules, or Weyl's constructions; $\forall \lambda \vdash k$, the functor $\mathbb{C}^n \mapsto W_{n;\lambda}$ is called a Schur functor); $W_{n;\lambda} \in \mathrm{Irrep}(GL_n)$, and $W_{n;\lambda} \ncong W_{n;\lambda'}$ for $\lambda \neq \lambda'$;
- $V_\lambda \otimes W_{n;\lambda}$ is
  - an S_k-rep. via $\sigma \cdot (v \otimes w) = (\sigma \cdot v) \otimes w$,
  - a GL_n-rep. via $g \cdot (v \otimes w) = v \otimes (g \cdot w)$.

Thus, there is a bijection between the "$\le n$ row part" of $\bigsqcup_k \mathrm{Irrep}(S_k)$, and a part of $\mathrm{Irrep}(GL_n)$. Not all irreps of GL_n are recovered this way. The dual irreps $W^*_{n;\lambda}$ are counter-examples. But, essentially, this is it: $\forall\, W \in \mathrm{Irrep}(GL_n)\ \exists\, k, d \in \mathbb{N}$ and $\lambda \vdash k$, $p(\lambda) \le n$, s.t.

$$W \cong W_{n;\lambda} \otimes \left(W_{n;1^n}^{\otimes d}\right)^*$$

Recall that, for $(V, \cdot) \in \mathrm{Rep}(G)$, its dual rep. V^* is the vector space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, with $(g \cdot f)(v) = f(g^{-1} \cdot v)$ for all $v \in V$, $f \in V^*$, $g \in G$.

Ex.: Take $n \ge 2$ and $k = 2$.

Then $V^{\otimes 2} = S^2(V) \oplus \Lambda^2(V)$ as an S_n-rep., and $(12) \in S_2$ acts
- on $S^2(V)$ by 1: $(12)(v_1 \otimes v_2 + v_2 \otimes v_1) = v_2 \otimes v_1 + v_1 \otimes v_2$;
- on $\Lambda^2(V)$ by -1: $(12)(v_1 \otimes v_2 - v_2 \otimes v_1) = v_2 \otimes v_1 - v_1 \otimes v_2$.

By the Schur-Weyl duality,

$V^{\otimes 2} \cong V^{tr} \otimes W_{n;2} \oplus V^{sgn} \otimes W_{n;1^2}$, and $(12) \in S_2$ acts

- on $V^{tr} \otimes W_{n;2}$ by 1;
- on $V^{sgn} \otimes W_{n;1^2}$ by -1.

But then $S^2(V) \cong V^{tr} \otimes W_{n;2} \cong W_{n;2}$ and $\Lambda^2(V) \cong V^{sgn} \otimes W_{n;1^2} \cong W_{n;1^2}$ } as GL_n-reps,

since $V^{tr} \cong V^{sgn} \cong \mathbb{C}$ as vector spaces, and GL_n acts only on the second factor of $V_\lambda \otimes W_{n;\lambda}$.

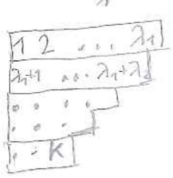
Conclusion: $S^2(V)$ & $\Lambda^2(V) \in \mathrm{Irrep}(S_n)$.

Ex0: Show this directly, without appealing to the SW duality.

Both the SW duality and the mutual centraliser property follow from the Double centraliser theorem for semisimple algebras. These are algebras A which, seen as A-reps (via multiplication on the left), are completely reducible. By Maschke's thm, $\forall$ finite group G, its group algebra $\mathbb{C}G$ is semisimple. So, the Double centraliser thm applies to $\mathbb{C}S_K$, and yields the SW duality.

We will give no further proof details here.
Instead, we will list some useful properties of Weyl reps:

① Explicit description: $\boxed{W_{n;\lambda} \cong C_\lambda \cdot V^{\otimes K}}$, for all $\lambda \vdash K$ with $p(\lambda) \leq K$,

where
- $C_\lambda = \left(\sum_{\sigma \in R(T_\lambda)} e_\sigma\right) \cdot \left(\sum_{\tau \in C(T_\lambda)} \mathrm{sgn}(\tau) e_\tau\right) \in \mathbb{C}S_K$, called the Young symmetriser for λ.
- $T_\lambda = $  $\in SYT_\lambda$
- $R(T)$ is the row group of T: $R(T) = \{\sigma \in S_K \mid \forall i,\ i \;\&\; \sigma(i) \text{ are in the same row of } T\}$
- $C(T)$ is the column group of T.

Ex.: • $\lambda = K$, $T_K = \boxed{1 \mid 2 \mid \cdots \mid K}$, $C_K = \left(\sum_{\sigma \in S_K} e_\sigma\right) \cdot e_{Id} = \sum_{\sigma \in S_K} e_\sigma$,

$\cdot\ W_{n;K} \cong \bigoplus_{1 \leq i_1 \leq \cdots \leq i_K \leq n} \mathbb{C}\, v_{i_1,\dots,i_K}$, where $v_{i_1,\dots,i_K} = \sum_{\sigma \in S_K} v_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes v_{i_{\sigma^{-1}(K)}}$

and $v_1, \dots, v_n$ is a basis of $V = \mathbb{C}^n$;

• $\lambda = 1^k$, $k \le n$, $T_{1^k} = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \vdots\\ \hline k\\ \hline\end{array}$, $c_{1^k} = e_{Id} \cdot (\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) e_\sigma) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) e_\sigma$,

$W_{n;1^k} \cong \bigoplus_{1 \le i_1 < \ldots < i_k \le n} \mathbb{C} w_{i_1,\ldots,i_k}$, where $w_{i_1,\ldots,i_k} = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) v_{i_{\sigma^{-1}(1)}} \otimes \ldots \otimes v_{i_{\sigma^{-1}(k)}}$.

• $\lambda = (2,1)$, $2 \le n$, $T_{2,1} = \begin{array}{|c|c|}\hline 1 & 2\\ \hline 3 \\ \cline{1-1}\end{array}$, $c_{2,1} = (e_{Id} + e_{(12)})(e_{Id} - e_{(13)}) = e_{Id} + e_{(12)} - e_{(13)} - e_{(132)}$

$W_{n;2,1}$ is the linear span of $v_{i_1} \otimes v_{i_2} \otimes v_{i_3} + v_{i_2} \otimes v_{i_1} \otimes v_{i_3}$

$- v_{i_3} \otimes v_{i_2} \otimes v_{i_1} - v_{i_2} \otimes v_{i_3} \otimes v_{i_1}$.

In fact, Young symmetrisers yield an alternative description of Specht reps: $V_\lambda \cong c_\lambda \cdot (\mathbb{C} S_k)$, for all $\lambda \vdash k$.

② Characters: $\boxed{\chi^{W_{n;\lambda}}(g) = S_{n;\lambda}(x_1,\ldots,x_n)}$ for all $\lambda \vdash k$ with $p(\lambda) \le n$, and $g \in GL_n$ with eigenvalues $x_1,\ldots,x_n$,

Here $S_{n;\lambda}$ is the Schur polynomial

$$S_{n;\lambda}(x_1,\ldots,x_n) = \frac{|x_j^{\lambda_i+n-i}|}{|x_j^{n-i}|} = \det \begin{pmatrix} x_1^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & \ldots & x_n^{\lambda_2+n-2} \\ \ldots & \ldots & \ldots \\ x_1^{\lambda_n} & \ldots & x_n^{\lambda_n} \end{pmatrix} \cdot \det \begin{pmatrix} x_1^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2} & \ldots & x_n^{n-2} \\ \ldots & \ldots & \ldots \\ x_1 & \ldots & x_n \\ 1 & \ldots & 1 \end{pmatrix}^{-1},$$

and we attach some 0's to λ to get a partition with n parts. From the form of $S_{n;\lambda}$, you probably guess that our formula for $\chi^{W_{n;\lambda}}$ follows from the Frobenius formula for χ^{V_λ}.
Recall that $|x_j^{n-i}| = \prod_{1 \le i < j \le n} (x_i - x_j)$. Since $|x_j^{\lambda_i+n-i}|$ changes sign when one exchanges some x_s & x_t, this polynomial is divisible by $|x_j^{n-i}|$, and their ratio is a symmetric polynomial in the x_j.

Ex: $S_{n;1^n}(x_1,\ldots,x_n) = \dfrac{|x_j^{n+1-i}|}{|x_j^{n-i}|} = \dfrac{x_1 \ldots x_n |x_j^{n-i}|}{|x_j^{n-i}|} = x_1 \ldots x_n$.

This is coherent with ②: $W_{n;1^n} = \mathbb{C} w_{1,\ldots,n}$, where

$w_{1,\ldots,n} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(k)}$,

so, if (v_i) is the basis for which $\rho(g)$ becomes $\begin{pmatrix} x_1 & * & & 0 \\ & x_2 & * & \\ & & x_3 & \ddots \\ 0 & & & x_n \end{pmatrix}$,

then $g \cdot w_{1,\ldots,n} = x_1 \ldots x_n w_{1,\ldots,n}$,

and $\chi^{W_{n;1^n}}(g) = x_1 \ldots x_n$ as predicted.

(Jordan normal form)

More generally, for $k \le n$, $S_{n;1^k} = \sum_{1 \le i_1 < \ldots < i_k \le n} x_{i_1} \ldots x_{i_k}$.

Crl: • the $S_{n;\lambda}$ are linearly independent polynomials

$\Rightarrow$ the $W_{n;\lambda}$ are "lin. indep. reps", i.e.,

$$\bigoplus_\lambda m_\lambda W_{n;\lambda} = \bigoplus_\lambda m'_\lambda W_{n;\lambda} \quad \text{iff} \quad \forall \lambda,\ m_\lambda = m'_\lambda ;$$

• $\boxed{\dim W_{n;\lambda}} = S_{n;\lambda}(1,\dots,1) = \lim\limits_{x\to 1} S_{n;\lambda}(1,x,x^2,\dots,x^{n-1}) = \lim\limits_{x\to 1} \dfrac{\prod_{1\le i<j\le n}(x^{\lambda_i+n-i} - x^{\lambda_j+n-j})}{\prod_{1\le i<j\le n}(x^{n-i} - x^{n-j})}$

$$\boxed{= \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j - i + j}{j-i}} .$$