Lecture 26:
More on the antiderivative–integral duality

Victoria LEBED

MA1S11A: Calculus with Applications for Scientists

December 11, 2017

1 Mean value theorem for integrals

With all the examples you have seen, you should be convinced that the antiderivative–integral duality given by the Fundamental Theorem of Calculus is useful in practice. Today we shall see where it comes from.

For this, we need an important property of definite integrals:

Theorem 7 (Mean Value Theorem for Integrals). Suppose that f is continuous on $[a, b]$. Then there is a point $c \in [a, b]$ satisfying

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

That is, the area between the segment $[a, b]$ and the curve $y = f(x)$ is the same as the area of a rectangle based on $[a, b]$ with height $f(c)$.
This value $f(c)$ is called the **mean value** of f on $[a, b]$.

1 Mean value theorem for integrals

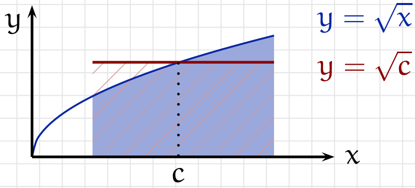
Theorem 7 (Mean Value Theorem for Integrals). Suppose that f is continuous on $[a, b]$. Then there is a point $c \in [a, b]$ satisfying

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Example. $\displaystyle\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}.$

According to the theorem, we have $\dfrac{14}{3} = \sqrt{c} \times (4-1)$ for some $c \in [1, 4]$.

In this case c is easy to find: $c = \left(\dfrac{14}{9}\right)^2 \approx 2.42 \in [1, 4]$.



2 Strong fundamental theorem of calculus

Now let us return to the antiderivative–integral duality:

Theorem 4 (The Fundamental Theorem of Calculus, part 1). If f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We will now prove a stronger version of it:

Theorem 8 (The Fundamental Theorem of Calculus, part 2). If f is continuous on $[a, b]$, then its antiderivative on $[a, b]$ can be defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Let us deduce Thm 4 from Thm 8. For this, in Thm 4 take the antiderivative F constructed in Thm 8. Since $F(a) = \int\limits_a^a f(t)\,dt = 0$, we obtain

$$F(b) - F(a) = F(b) = \int_a^b f(t)\,dt,$$

as desired.

2 Strong fundamental theorem of calculus

Theorem 8 (The Fundamental Theorem of Calculus, part 2). If f is continuous on $[a, b]$, then its antiderivative on $[a, b]$ can be defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Proof. We need to check that F is an antiderivative of f: $F' = f$, i.e.,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

We have

$$\begin{aligned}
\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] &= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \\
&\overset{\text{Thm 2}}{=} \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \overset{\text{MVT}}{=} \lim_{h\to 0} \frac{f(c(x,h))h}{h} \\
&= \lim_{h\to 0} f(c(x,h)) = f(x),
\end{aligned}$$

as desired. Here $c(x, h)$ is the point between x and $x + h$ given by MVT (the Mean Value Theorem for Integrals). Since it gets closer to x when h gets smaller, and since f is continuous, $f(c(x, h)) \underset{h\to 0}{\to} f(x)$. $\square$

2 Strong fundamental theorem of calculus

Theorem 8 (The Fundamental Theorem of Calculus, part 2). If f is continuous on $[a, b]$, then its antiderivative on $[a, b]$ can be defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Remark. Choosing any point $c \in [a, b]$, we get another antiderivative

$$\widetilde{F}(x) = \int_c^x f(t)\,dt$$

of f. Indeed, the two functions differ by a constant:

$$\widetilde{F}(x) - F(x) = \int_c^x f(t)\,dt - \int_a^x f(t)\,dt \overset{\text{Thm 2}}{=} \int_c^a f(t)\,dt.$$

3 Derivatives, antiderivatives, integrals

Let us summarise what we know about the three basic notions of calculus:

derivative f'	antiderivative, or indefinite integral $\int f = F(x) + C$	definite integral $\int_a^b f$
differentiation	symbolic integration	numerical integration
tangents, rate of change (e.g. velocity)		area, average value

The connection between the first two columns is obvious: by definition,

$$(\textstyle\int f)' = f, \qquad \textstyle\int (f') = f + C.$$

The connection between the last two columns is very deep, and given by the Fundamental Theorem of Calculus:

$$\textstyle\int_a^b f = [F]_a^b, \qquad F(x) = \textstyle\int_a^x f.$$

This theorem revolutionised mathematics and physics: before it, differentials and integrals had been treated separately (and somewhat painfully!) for centuries.

4 Integral calculus: how-tos

✓ Computing indefinite integrals / antiderivatives:

- reduce to simpler integrals using linearity;
- simplify using u-substitutions;
- use (repeated) integration by parts to simplify or obtain a relation of type $\int f = \text{smth} - \int f$;
- look up in the table of derivatives.

✓ Computing definite integrals:

- use the Fundamental Theorem of Calculus (compute an antiderivative and subtract its values);
- if an antiderivative is difficult to find, apply a u-substitution to (a part of) the integral;
- (usually quite tedious) find the limit of Riemann sums.

✓ Computing areas:

- compute the areas below the curves involved using definite integrals, then add / subtract them.