~~22~~ 25. Representation theory of Braid groups

Let us summarise what we have learned about the reps of a finite group G in the main part of this course:

1) G has a finite nb of irreps, $\# \mathrm{Irrep}(G) = \# \mathrm{Conj}(G) =: k$,
2) any rep. of G is a direct sum of irreps with uniquely determined multiplicities;

   that is, there is an isomorphism of monoids

   $$\mathrm{Rep}(G) \cong \mathbb{N}_0^{\# \mathrm{Conj}(G)}$$
   $$V = m_1 V_1 \oplus \ldots \oplus m_k V_k \longmapsfrom (m_1, \ldots, m_k)$$

   In other words, $\mathrm{Rep}(G)$ is parametrised by the infinite countable set $\mathbb{N}_0^k$.

   where $\mathbb{N}_0^k$ is the direct product of k copies of the monoid $\mathbb{N}_0 = (\mathbb{N} \cup \{0\}, +)$,

   and $\mathrm{Irrep}(G) = \{V_1, \ldots, V_k\}$;
3) the multiplicities m_i can be computed using characters:

   $m_i = (\chi^V, \chi^{V_i})$
4) more generally, characters provide a powerful tool for studying reps:
   - $V \cong W \Leftrightarrow \chi^V = \chi^W$
   - $V \in \mathrm{Irrep}(V) \Leftrightarrow (\chi^V, \chi^V) = 1$
5) to describe $\mathrm{Irrep}(G)$ explicitly, or at least construct the character table of G, one uses a mixture of techniques:
   - determine degree 1 reps by direct computations;
   - use an interpretation of your group in terms of symmetries of geometric objects;
   - decompose into irreducibles tensor products of known irreps;
   - use restriction & induction of reps;
   - if you know a group morphism $\varphi : G \to H$ from your group to a group whose reps you understand, look at the image of the "pull-back" map $\varphi^* : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$;
   - check Schur's orthonormality relations for the (rescaled) character table;
   - use relations between the degrees of irreps:

     $\sum_i (\dim_{\mathbb{C}} V_i)^2 = \# G$,

     $\dim_{\mathbb{C}} V_i$ divides $\# G$ (not seen in class), etc.

1

We managed to describe completely the reps of certain classes of groups:

→ abelian groups;
→ symmetric groups S_n;
→ alternating groups A_n;
→ dihedral groups D_{2n}
(the description obtained in HW2 for D_8 & in Q2 for D_{10} generalises to all even & odd n respectively);
→ quaternion group Q.

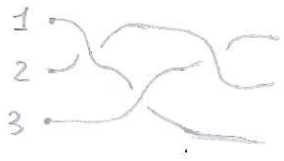
As you might guess, this in the tip of the iceberg: there are a lot of techiques and results we have not even touched. Also, we did not talk about multiple applications of representation theory – for instance, to the classification of finite simple groups of to Galois theory.

In the next block of lectures we will look at an important family of infinite groups – braid groups. Even though they are closed relatives of the S_n, their rep. theory is much more wild – and less understood.

As explained in Lecture 1, there might be several ways to define the same group, suitable for different purposes. This is the case for braid groups.

① Topological definition: group $\mathcal{B}_n$

• elements: geometrical braids on n strands up to isotopy



$=: B \in \mathcal{B}_3$

• group operation: horizontal stacking

• neutral element:

• inverse: mirror image

$B^{-1} =$ 

Exo: For these data, check the axioms from the definition of a group.

we gave a vague but transparent version of this definition. With some more work it can be made perfectly rigourous. As our main definition, we'll choose another one. It will be less intuitive, but better adapted for studying representations.

② Algebraic definition by generators & relations:

$$\boxed{B_n = \left\langle \sigma_1,\dots,\sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j=\sigma_j\sigma_i \text{ for } |i-j|>1, & 1\le i,j\le n-1 \;(1) \\ \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}, & 1\le i\le n-2 \;(2) \end{array}\right\rangle}$$

The elements of B_n are words in $\sigma_i^{\pm 1}$, considered up to relations (1) & (2) & (3)

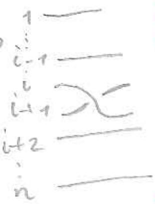
$$\boxed{\sigma_i\sigma_i^{-1}=\sigma_i^{-1}\sigma_i=1 \;(3)}$$

Ex.: $\sigma_1\sigma_2^{-1}\sigma_1 \in B_3$,

$$\sigma_1\sigma_2^{-1}\sigma_1 \overset{(2)}{=} \sigma_1\underbrace{\sigma_2^{-1}(\sigma_2}\sigma_1\sigma_2)(\sigma_1^{-1}\sigma_2^{-1}\underbrace{\sigma_1^{-1})\sigma_1} \overset{(2)}{=} \sigma_1^2\sigma_2\sigma_1^{-1}\sigma_2^{-1}.$$

Thm 13 (Artin '25): A group isomorphism can be defined by

$\Phi : B_n \to \mathcal{B}_n$

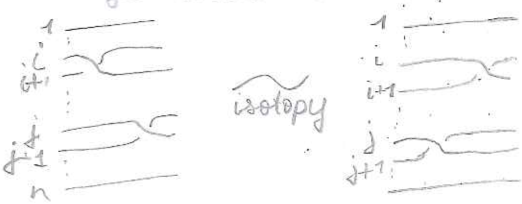
$\sigma_i \mapsto$ 

Ex.: $\Phi(\sigma_1\sigma_2^{-1}\sigma_1) = \mathcal{B}$ (see page 2)

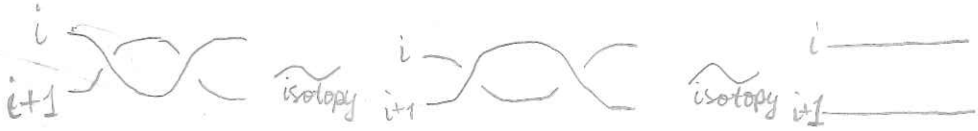
☐ 1) Check that Φ is well defined (then it is automatically a group homomorphism).

That is, check that Φ sends the LHS & the RHS of (1)–(3) to isotopic geometrical braids.

For (1), with $j>i+1$:

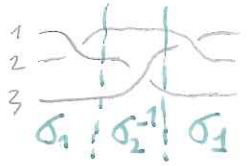


For (2):

For (3): 

(The upper & lower horizontal strands are omitted in our pictures).

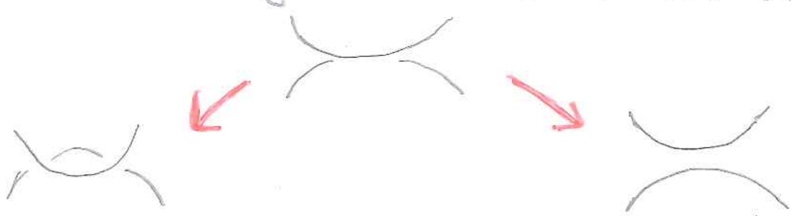
2) Φ is surjective since every braid, eventually after being slightly deformed, decomposes into "elementary blocks" corresponding to $\sigma_i^{\pm 1}$:



3) For injectivity, we give only the idea of the proof. While being deformed, a braid does not change its elementary block decomposition unless it goes through a "singular state", which does not decompose into blocks. There are three types of singular states:

Slight deformation of each of these states yields two different desingularisations, which correspond to the LHS & RHS of the defining relations (1)-(3) of B_n!

For example,



One should also consider the singular states of types (1) & (2) with some crossings replaced with the opposite ones. For those, one gets relations in B_n which are consequences of (1)-(3).

For example, for $|i-j|>1$, $\sigma_i\sigma_j^{-1}=\sigma_j^{-1}\sigma_i \Leftarrow \sigma_i\sigma_j=\sigma_j\sigma_i$
$\& \ \sigma_j\sigma_j^{-1}=\sigma_j^{-1}\sigma_j=1$ $\boxtimes$

In what follows, we will use the group B_n only, but will keep in mind braid diagrams (the group $\mathcal{B}_n$) for developing a good intuition of that group.

Rmk: In fact, braid groups appear in an extraordinary variety of mathematical contexts. In particular, they have several other equivalent definitions:

→ as subgroups of the automorphism groups of free groups;
→ as the fundamental groups of the configuration space of n distinct points in the plane;
→ as the mapping class groups of the complement of n points in a disk.

For studying representations, it is particularly important to understand relations between the B_n & other familiar groups:

(a) $\boxed{\begin{array}{l}\pi_S: B_n \twoheadrightarrow S_n \\ \sigma_i \mapsto (i,i+1)\end{array}}$

notation: $\twoheadrightarrow$ means "surjective group homomorphism"

$\hookrightarrow$ means "injective group hom."

Crl: • B_n is not abelian for $n\geq 3$
$\Uparrow$
S_n is not ab for $n\geq 3$: $(i,i+1)(i+1,i+2)=(i,i+1,i+2)\neq(i,i+2,i+1)=(i+1,i+2)(i,i+1)$

• braids can be seen as "permutations with history", which explains their appearance in quantum physics & quantum computing.

(b) $\boxed{\begin{array}{l}\pi_{\mathbb{Z}}: B_n \twoheadrightarrow (\mathbb{Z},+) \\ \sigma_i \mapsto 1\end{array}}$

• for $n=2$, it is a group isomorphism: $\boxed{B_2 \cong \mathbb{Z}}$

Crl: B_n is infinite $\Leftarrow$ $\mathbb{Z}$ is infinite.

(c) $\boxed{\begin{array}{l}\iota: B_n \hookrightarrow B_{n+1} \\ \sigma_i \mapsto \sigma_i\end{array}}$

(d) $\boxed{\begin{array}{l}\pi_B: B_4 \twoheadrightarrow B_3 \\ \sigma_1,\sigma_3 \mapsto \sigma_1 \\ \sigma_2 \mapsto \sigma_2\end{array}}$

This group morphism is proper to B_4, and doesn't extend to other B_n.
It is related to the group morphism $S_4 \twoheadrightarrow S_3$ seen in T2, which, again, doesn't work for other n.

Recall that any group morphism $\mathcal{G} \xrightarrow{\varphi} H$ induces a "pull-back" map $\operatorname{Rep}(H) \xrightarrow{\varphi^*} \operatorname{Rep}(\mathcal{G})$, which is injective on irreps if φ is surjective:

$$\boxed{\mathcal{G} \overset{\varphi}{\twoheadrightarrow} H \quad \Rightarrow \quad \operatorname{Irrep}(\mathcal{G}) \overset{\varphi^*}{\hookleftarrow} \operatorname{Irrep}(H).}$$

So the above relations give for B_n a source of:

(a) irreps with a finite image $\rho(B_n)$;

(b) degree 1 irreps:

$$\operatorname{Irrep}(\mathbb{Z}) = \underset{\nearrow \atop \text{degree } 1}{\operatorname{Irrep}_1(\mathbb{Z})} = \left\{ \xi_t : \boxed{\begin{matrix} \mathbb{Z} \to \mathbb{C}^* \\ k \mapsto t^k \end{matrix}} \;\middle|\; t \in \mathbb{C}^* \right\}$$

$\theta_t := \pi_{\mathbb{Z}}^* \xi_t : \boxed{B \mapsto t^{\pi_{\mathbb{Z}}(B)}}$

Ex.: $\theta_t(\sigma_i) = t$, $\theta_t(\sigma_1 \sigma_2^{-1}) = 1$, $\theta_t(\sigma_i^k) = t^k$.

(d) special reps for B_4.

Next, we will describe an important family of B_n-reps different from those in (a)-(d).

Prop. 3.2 (Burau '36): For any $n \geq 2$ & $t \in \mathbb{C}^*$, a representation $V_t^B = (\mathbb{C}^n, \rho_t)$ of B_n can be defined by putting

$$\boxed{\underbrace{\rho_t(\sigma_i) = I_{i-1} \oplus M_t \oplus I_{n-i-1}}_{\text{block-diagonal matrix}}}, \quad \text{where } M_t = \begin{pmatrix} 0 & 1 \\ t & 1-t \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \boxed{\begin{matrix} 0 & 1 \\ t & 1-t \end{matrix}} & \\ & & & & 1 \\ 0 & & & & \ddots \quad 1 \end{pmatrix}$$

□ The matrix M_t, hence $\rho_t(\sigma_i)$, is invertible for $t \neq 0$.
It remains to check that the map ρ_t is compatible with the relations (1) & (2) defining B_n; it thus defines a group homomorphism $B_n \to GL_n(\mathbb{C}) = \operatorname{Mat}^*_{n\times n}(\mathbb{C}) = \operatorname{Aut}_{\mathbb{C}}(\mathbb{C}^n)$, hence a B_n-rep.

For (1), ρ_t sends both sides to $I_{i-1} \oplus M_t \oplus I_{j-i-2} \oplus M_t \oplus I_{n-j-1}$.
We supposed here $j \geq i+2$.

For (2), one has to check the matrix equality

$$\begin{pmatrix} 0 & 1 & 0 \\ t & 1-t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t & 1-t \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ t & 1-t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t & 1-t \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ t & 1-t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t & 1-t \end{pmatrix},$$

which is straightforward. $\boxtimes$

The V_t^B are called Burau reps of B_n.

In order to describe their properties, we should study the matrix $M_t = \begin{pmatrix} 0 & 1 \\ t & 1-t \end{pmatrix}$.

- Its eigenvalues are 1 and $-t$, since

  $\det(M_t - xI_2) = \det\begin{pmatrix} -x & 1 \\ t & 1-t-x \end{pmatrix} = x(x+t-1) - t = (x-1)(x+t)$.

- The corresponding eigenvectors are $\binom{1}{1}$ and $\binom{1}{-t}$.

  Indeed, $\begin{pmatrix} 0 & 1 \\ t & 1-t \end{pmatrix}\binom{a}{b} = \binom{a}{b} \Leftrightarrow \binom{b}{ta+(1-t)b} = \binom{a}{b} \Leftrightarrow a = b$,

  and similarly for $-t$.

- Jordan normal form: $\begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix}$ when $t \neq -1$,

  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ when $t = 1$, since in this case there is only 1 eigenvalue $(=1)$ with 1 eigenvector, up to a scalar multiple.

- $\det(M_t) = -t$.

Properties of V_t^B:

① $\chi^{V_t^B}(\sigma_i) = n-1-t$

$\square$ eigenvalues of $\rho_t(\sigma_i)$: $\begin{cases} 1, & n-1 \text{ times}; \\ -t, & 1 \text{ time}. \end{cases}$ $\boxtimes$

Crl: $V_{t_1} \not\cong V_{t_2}$ for $t_1 \neq t_2$.

Rmk: $V \cong W \Rightarrow \chi^V = \chi^W$ for any G and any $V, W \in Rep(G)$, whereas the "$\Leftarrow$" direction works only for finite G.

Thus, we have a family of B_n-reps indexed by a continuous uncountable parameter t, while for a finite group G the whole $Rep(G)$ is indexed by the discrete countable set $\mathbb{N}_0^k$, $k = \# Conj(G)$.

② $\chi^{V_t^B}(\sigma_i^k) = n-1+(-t)^k$ for all $k \in \mathbb{Z}$.

$\square$ eigenvalues of $\rho_t(\sigma_i^k)$: $\begin{cases} 1, & n-1 \text{ times} \\ (-t)^k, & 1 \text{ time}. \end{cases}$ $\boxtimes$

- Crl: For t not root of unity, $\rho_t(B_n)$ is an infinite subgroup of $GL_n(\mathbb{C})$. In particular, $V_t^B \notin \pi_s^*(Rep(S_n))$.

③ $V_0 := \mathbb{C}\begin{pmatrix}1\\1\\ \vdots \\1\end{pmatrix}$ is a subrep. of V_t^B, with $V_0 \cong V^{tr}$. For $n \geq 3$, it is the only degree 1 subrep.

$\square$ $M_t\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \forall i,\ \rho_t(\sigma_i)\begin{pmatrix}1\\ \vdots\\1\end{pmatrix} = \begin{pmatrix}1\\ \vdots \\1\end{pmatrix} \Rightarrow V_0$ is a subrep, with $V_0 \cong V^{tr}$.
Now, suppose $n \geq 3$, and take a subrep. $\mathbb{C}\begin{pmatrix}a_1\\ \vdots\\a_n\end{pmatrix}$, with $\begin{pmatrix}a_1\\ \vdots\\a_n\end{pmatrix} \neq \begin{pmatrix}0\\ \vdots\\0\end{pmatrix}$.
For some i, $a_i \neq 0 \Rightarrow \begin{pmatrix}a_{i-1}\\a_i\end{pmatrix}$ and $\begin{pmatrix}a_i\\a_{i+1}\end{pmatrix}$ are, when they exist, non-zero eigenvectors for $M_t \Rightarrow a_{i-1} \neq 0,\ a_{i+1} \neq 0$.
Iterating this argument, one gets $a_j \neq 0$ for all j.
Now, $\forall i\ \exists \lambda_i \in \mathbb{C}$ s.t. $\rho_t(\sigma_i)\begin{pmatrix}a_1\\ \vdots\\a_n\end{pmatrix} = \lambda_i\begin{pmatrix}a_1\\ \vdots\\a_n\end{pmatrix}$. For $j \notin \{i, i+1\}$, this means (here we suppose $n \geq 3$)
$a_j = \lambda_i a_j$, hence $\lambda_i = 1$ (since $a_j \neq 0$). Thus, $\forall i$, $\begin{pmatrix}a_i\\a_{i+1}\end{pmatrix}$ is an eigenvector of M_t with the eigenvalue $1 \Rightarrow a_i = a_{i+1}$. So, $\mathbb{C}\begin{pmatrix}a_1\\ \vdots\\a_n\end{pmatrix} = \mathbb{C}\begin{pmatrix}1\\ \vdots\\1\end{pmatrix}$ $\boxtimes$

④ $V_t^{rB} := \left\{ \begin{pmatrix}v_1\\ \vdots\\v_n\end{pmatrix} \in \mathbb{C}^n \mid t^{n-1}v_1 + \dots + t v_{n-1} + v_n = 0 \right\}$ is a degree $n-1$ subrep. of V_t^B, called a reduced Burau rep. of B_n.

(a) If $t^{n-1} + \dots + 1 \neq 0$, then $V_t^B = V_t^{rB} \oplus V_0$ (decomposition of B_n-reps).

(b) If $t^{n-1} + \dots + 1 = 0$, then $\underset{n}{V_t^B} \supset \underset{n-1}{V_t^{rB}} \supseteq \underset{1}{V_0}$ ($\dim_{\mathbb{C}}$: n, $n-1$, 1), and V_t^{rB} has no B_n-invariant complement in V_t^B.

$\square$ $(t\ 1)\begin{pmatrix}0 & 1\\ t & 1-t\end{pmatrix} = (t\ 1) \Rightarrow \forall i,\ (t^{n-1} \dots t\ 1)\rho_t(\sigma_i) = (t^{n-1} \dots t\ 1)$

Now, V_t^{rB} is

- clearly a linear subspace of V_t^B of dim. $n-1$, with a basis $\begin{pmatrix}1\\-t\\0\\ \vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-t\\0\\ \vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\ \vdots\\0\\1\\-t\end{pmatrix}$;
- B_n-invariant, since $\begin{pmatrix}v_1\\ \vdots\\v_n\end{pmatrix} \in V_t^{rB} \Rightarrow (t^{n-1} \dots t\ 1)\begin{pmatrix}v_1\\ \vdots\\v_n\end{pmatrix} = 0 \Rightarrow \forall i,$
$(t^{n-1} \dots t\ 1)\rho_t(\sigma_i)^{\pm 1}\begin{pmatrix}v_1\\ \vdots\\v_n\end{pmatrix} = (t^{n-1} \dots t\ 1)\begin{pmatrix}v_1\\ \vdots\\v_n\end{pmatrix} = 0 \Rightarrow \forall i,\ \sigma_i^{\pm 1} \cdot \begin{pmatrix}v_1\\ \vdots\\v_n\end{pmatrix} \in V_t^{rB}$.

so, V_t^{rB} is indeed a sub-rep.

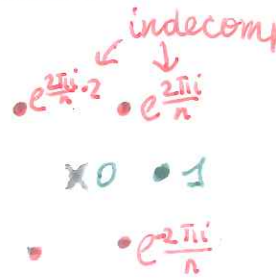
V_t^{rB} has a B_n-invar. complement $\iff$ $V_t^B = V_t^{rB} \oplus V$.

$\dim_{\mathbb{C}}$: n $\quad$ $n-1$ $\quad$ 1

If $n \geq 3$, then, by ③, the only possibility is $V_t^B = V_t^{rB} \oplus V_0$, which is equivalent to $V_t^{rB} \cap V_0 = \{0\}$, i.e., $\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \notin V_t^B$, which means $t^{n-1}+\dots+1 = (t^{n-1} \dots t\ 1)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} \neq 0$.

If $n=2$, then, from the Jordan normal form of $\rho_t(\sigma_1) = M_t$, one sees that V_t^B is decomposable when $t+1 \neq 0$ and indecomposable otherwise ⊠.

Rmk:
- In fact, in case (a) V_t^{rB} is irreducible, and in case (b) the quotient rep. V_t^{rB}/V_0 is irreducible, which completely solves the irreducibility question for V_t^B.
- In our rep. family parametrised by $t \in \mathbb{C}^*$, the reps corresponding to n^{th} rooots of unity $t \neq 1$ exhibit the special indecomposable behaviour, since $t^{n-1}+\dots+t+1 = \frac{t^n-1}{t-1}$.



- The rep. V_1^B is also very special:

$$\boxed{V_1^B \cong \pi_s^*(V^{perm}),\quad V_1^{rB} \cong \pi_s^*(V^{st})}$$

□ Indeed, in $V^p = \bigoplus_{j=1}^n \mathbb{C}e_j$, $(i,i+1)\cdot e_j = \begin{cases} e_j, & j \notin \{i, i+1\}, \\ e_{i+1}, & j = i, \\ e_i, & j = i+1, \end{cases}$

so, in the basis (e_j), $\rho(i,i+1)$ has the matrix $I_{i-1} \oplus \begin{pmatrix}0&1\\1&0\end{pmatrix} \oplus I_{n-i-1}$, which is precisely the matrix $\rho_1(\sigma_i)$. Since $\pi_s(\sigma_i) = (i,i+1)$, this yields $V_1^B \underset{(\rho_1 = \rho\pi_s)}{\cong} \pi_s^*(V^p)$. Further, V_1^{rB} is the complement of $V_0 \cong V^{tr}$ in V_1^B, and V^{st} is the compement of $\mathbb{C}(e_1+\dots+e_n) \cong V^{tr}$ in V^p, so $V_1^{rB} \cong \pi_s^*(V^{st})$. ⊠

⑤ Is V_t^B faithful? That is, is $\rho_t : B_n \to GL_n(\mathbb{C})$ injective?

Yes for
- $n=2$ (easy: $B_2 \cong \mathbb{Z}$)
- $n=3$ (proved in 1969)

No for
- $n \geq 9$ ('91)
- $n \geq 6$ ('93; a counter-example with 76 crossings)
- $n \geq 5$ ('99; an even more monstrous counter-example, found by a computer).

Open question for $n=4$!!!

You see a beautiful example of an easily stated problem which has been open for almost a century.

Rmk: Faithful reps of the B_n were constructed for all n around 2000. Their degree is $\frac{n(n-1)}{2}$. This allowed to identify the B_n with subgroups of general linear groups, and deduce numerous corollaries about the structure of the B_n.

The main question of the rep. theory of the B_n is:

How much do we know about $\mathrm{Rep}(B_n)$?

The answer is: very little...

We will give several partial results. The first one is given without proof, for general culture purposes.

$\boxed{n-1}$ Thm (Formanek '94): For $n \neq 4, 5, 6$,

$$\underbrace{\mathrm{Irrep}_{\leq n-1}(B_n)}_{\text{of degree } \leq n-1} \subseteq \mathrm{Irrep}_1(B_n) \cup \mathrm{Irrep}_1(B_n) \otimes \{ V_t^{rB} \text{ or } V_t^{rB}/V_0 \mid t \in \mathbb{C}^* \}$$

This means that, up to degree $n-1$, the only irreps of B_n, $n \neq 4,5,6$, are degree 1 reps, irred. quotients of reduced Burau reps, and tensor products thereof.

Rmk: • Since 1994, people classified $\mathrm{Irrep}_{\leq n^2}(B_n)$, but the whole $\mathrm{Rep}(B_n)$ remains a mystery.

- For $n = 4, 5, 6$, there are more "small" irreps, coming from the following sources:
  - $\pi_B : B_4 \twoheadrightarrow B_3$
  - $\pi_S : B_4 \twoheadrightarrow S_4$
    $\dim_{\mathbb{C}} V_{(2,2)} = 2$
  - $B_5 \hookrightarrow B_6 \xrightarrow{\pi_S} S_6$
    $\dim_{\mathbb{C}} V_{(3,3)} = \dim_{\mathbb{C}} V_{(2,2,2)} = 5$

Exo: Using the Hook length formula, prove that
$\dim_{\mathbb{C}} V_\lambda < n \iff \lambda \in \{ (n), (1^n), (n-1,1), (2,1^{n-2}), (2,2), (2,2,2), (3,3) \}$

	(n)	(1^n)	$(n-1,1)$	$(2,1^{n-2})$	$(2,2)$	$(2,2,2)$	$(3,3)$
	V^{tr}	V^{sgn}	V^{st}	$(V^{st})'$			
$\dim_{\mathbb{C}}$:	1	1	$n-1$	$n-1$	2	5	5

We will next describe $Rep_1(B_n)$ and $Rep_2(B_n)$, this time with proofs.

$\boxed{1}$ Prop. 33: $\boxed{Rep_1(B_n) = \{ \theta_t \mid t \in \mathbb{C}^* \}}$ (see page 6 for the def^n of θ_t)

$\square$ We know that $\theta_t = \pi_{\mathbb{Z}}^*(\xi_t)$, where the map $\pi_{\mathbb{Z}}^* : Irrep(\mathbb{Z}) \hookrightarrow Irrep(B_n)$ is injective. To finish the proof,
($Irrep(\mathbb{Z}) = Rep_1(\mathbb{Z}) = \{\xi_t \mid t \in \mathbb{C}^*\}$)
observe that $\forall\, \rho : B_n \xrightarrow[group]{} \mathbb{C}^*$,
$\rho(\sigma_i) = \rho(\sigma_1)$ for all i (by Lemma 34),
so the reps $(\mathbb{C}, \rho)$ and $\theta_{\rho(\sigma_1)}$ take the same value on all the generators σ_i of B_n, hence coincide. $\boxtimes$

Lemma 34: $\forall\, 1 \le i, j \le n-1$, $\sigma_i \sim \sigma_j$ (i.e., are conjugate in B_n).

$\square$ $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \Rightarrow (\sigma_i \sigma_{i+1})^{-1} \sigma_{i+1} (\sigma_i \sigma_{i+1}) = \sigma_i \Rightarrow \sigma_i \sim \sigma_{i+1}$.
Iterating this argument, one gets $\sigma_i \sim \sigma_j$ for all i, j. $\boxtimes$

Rmk: Prop. 33 also follows from $Ab(B_n) \cong \mathbb{Z}$ (see HW1).

Rmk: $\forall\, t, s \in \mathbb{C}^*$, $\theta_t \theta_s = \theta_{ts}$. Thus, $\boxed{\begin{array}{c} \mathbb{C}^* \to (Rep_1(B_n), \otimes) \\ t \mapsto \theta_t \end{array}}$ is a group isomorphism.

$\boxed{2}$ We will next look at $\overline{\mathrm{Rep}}_2(B_3) = \{\rho : B_3 \to SL_2(\mathbb{C})\}/\sim$,

where: • $SL_2(\mathbb{C}) = \{M \in Mat_{2\times2}(\mathbb{C}) \mid \det M = 1\}$

• $\rho \sim \rho'$ means: $\exists\, M \in GL_2(\mathbb{C})$ s.t. $\forall \beta \in B_3$, $\rho(\beta) = M\rho(\beta')M^{-1}$

(It is the usual notion of isomorphism of B_3-reps).

• $B_3 = \langle a, b \mid aba = bab\rangle$ (We use notations $a = \sigma_1$, $b = \sigma_2$ for a better readability).

Let us explain why we study $\overline{\mathrm{Rep}}_2$ instead of the usual Rep_2. On the one hand, Rep_2 can be easily recovered from $\overline{\mathrm{Rep}}_2$:

Lemma 35: $\forall\, (\mathbb{C}^2, \rho) \in \mathrm{Rep}_2(B_3)$, $\rho \otimes \theta_\alpha \in \overline{\mathrm{Rep}}_2(B_3)$, where $\alpha^{-2} = \det(\rho(a))$.

Crl: $\mathrm{Rep}_2(B_3) = \overline{\mathrm{Rep}}_2(B_3) \otimes \mathrm{Rep}_1(B_3)$.

Proof of Crl: □ $(\rho\otimes\theta_\alpha)\otimes\theta_{\alpha^{-1}} = \rho\otimes(\theta_\alpha\otimes\theta_{\alpha^{-1}}) = \rho\otimes\theta_1 = \rho$, so any degree 2 rep. ρ is a tensor product of a "normalised" degree 2 rep. and a degree 1 rep. (In fact, $\forall \rho$ has precisely 2 such tensor decompositions, corresponding to 2 different α satisfying $\alpha^{-2} = \det\rho(a)$.) ☒

Proof of L.35: □ $\det((\rho\otimes\theta_\alpha)(a)) = \det(\alpha\rho(a)) = \alpha^2\det(\rho(a)) = 1$.
By Lemma 34, this implies $\det((\rho\otimes\theta_\alpha)(b)) = 1$
Since det is a multiplicative function, one concludes that $\det((\rho\otimes\theta_\alpha)(\beta)) = 1$ for all $\beta \in B_n$. ☒

On the other hand, $\overline{\mathrm{Rep}}_2(B_3)$ is much easier to work with, due to

Lemma 36: For $A, B \in SL_2(\mathbb{C})$,
(a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
(b) $\mathrm{tr}(A^{-1}) = \mathrm{tr}(A)$
(c) $\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

□ (a) is obvious.

(b): $\left.\begin{array}{l}\text{eigenvalues of } A: \lambda \text{ and } \frac{1}{\lambda}\\ \text{eigenvalues of } A^{-1}: \lambda^{-1} \text{ and } \lambda\end{array}\right\} \Rightarrow \mathrm{tr}(A) = \lambda + \frac{1}{\lambda} = \mathrm{tr}(A^{-1})$.

(c): after a basis change, A writes as $\begin{pmatrix}\lambda & a\\ 0 & \lambda^{-1}\end{pmatrix}$, with $a \in \{0,1\}$. (Jordan normal form), and B as $\begin{pmatrix} b & c\\ d & e\end{pmatrix}$. So, A^{-1} writes as $\begin{pmatrix}\lambda^{-1} & -a\\ 0 & \lambda\end{pmatrix}$, and

$\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}\begin{pmatrix}\lambda b + ad & \lambda c + ae\\ \lambda^{-1}d & \lambda^{-1}e\end{pmatrix} + \mathrm{tr}\begin{pmatrix}\lambda^{-1}b - ad & \lambda^{-1}c - ae\\ \lambda d & \lambda e\end{pmatrix} = (\lambda b + ad + \lambda^{-1}e) +$
$+ (\lambda^{-1}b - ad + \lambda e) = (\lambda + \lambda^{-1})(b + e) = \mathrm{tr}(A)\,\mathrm{tr}(B)$. ☒

Thm 14: (a) The map

$$\mathrm{Ch} : \overline{\mathrm{Rep}}_2(B_3) \twoheadrightarrow \{(x,y)\in\mathbb{C}^2 \mid (y-2)(x^2-y-1)=0\}$$

$$\rho \longmapsto (x := \chi^{\rho}(a),\ y := \chi^{\rho}(ab^{-1}))$$

is well defined and surjective;

(b) The map Ch restricts to a bijection

$$\overline{\mathrm{Rep}}_2^{dec}(B_3) \overset{1:1}{\longleftrightarrow} \{(x,y)\in\mathbb{C}^2 \mid y-2=0\} \overset{1:1}{\longleftrightarrow} \mathbb{C}$$

$$(x,y) \longmapsto x$$

$=$ {decomposable reps in $\overline{\mathrm{Rep}}_2(B_3)$}.

(c) The map Ch restricts to a surjection

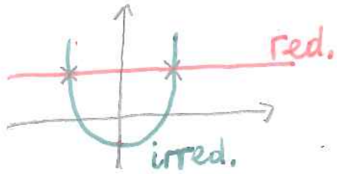
$$\overline{\mathrm{Rep}}_2^{rB}(B_3) \twoheadrightarrow \{(x,y)\in\mathbb{C}^2 \mid x^2-y-1=0\} \overset{1:1}{\longleftrightarrow} \mathbb{C}$$

$$(x,y) \longmapsto x$$

$=$ {normalised V_t^{rB} | $t\in\mathbb{C}^*$}

(d) $\forall\, V, W \in \overline{\mathrm{Rep}}_2(B_3)$, $\mathrm{Ch}(V)=\mathrm{Ch}(W) \Leftrightarrow \chi^V=\chi^W$.

the character variety of B_3



×: (1) reducible but indecomposable
(2) decomposable
(≥ 2 pre-images)

Crl: $\mathrm{Ch}^{-1}(\sqrt{3},2)$ contains two non-isomorphic reps with the same character

□ By (b), $\exists\, \rho\in\overline{\mathrm{Rep}}_2^{dec}(B_3)$ with $\mathrm{Ch}(\rho)=(\sqrt{3},2)$.

By (c), $\exists\, t\in\mathbb{C}^*$ with $\mathrm{Ch}(V_t^{rB}\otimes\theta_\alpha)=(\sqrt{3},2)$, and $\alpha^{-2}=-t$. Indeed, $(\sqrt{3})^2-2-1=0$.

By (d), $\chi^\rho=\chi^V$, $V:=V_t^{rB}\otimes\theta_\alpha$.

Since for $n=3$ reduced Burau reps are indecomposable ($\Leftarrow V_t^{rB}$ has at most 1 degree 1 subrep.),

V is also indecomposable. But ρ is decomposable.

So $V \not\cong (\mathbb{C}^2,\rho)$ ⊠.

Rmk: • From the proof of thm 14, it will be clear that

$\mathrm{Ch}(V_t^{rB}\otimes\theta_\alpha)=(\pm\sqrt{3},2)$ precisely when $t^2+t+1=0$, i.e.,

when V_t^{rB} is reducible (but indecomposable).

This explains why this point is so special.

• With some more work, one shows:

• ρ is reducible $\Leftrightarrow \chi^\rho(ab^{-1})=2$;

• For $V, W\in\overline{\mathrm{Irrep}}_2(B_3)$, $V\cong W \Leftrightarrow \chi^V=\chi^W$

Therefore, $\overline{\mathrm{Irrep}}_2(B_3)=\{$normalised $V_t^{rB} \mid t^2+t+1\neq 0\}$.

• More generally, $\overline{\mathrm{Rep}}_2(G)$ can be described in terms of an algebraic variety for a wide class of groups.

Proof of thm 14:

□ (a) By L.36, $\chi^\rho(ab) = \chi^\rho(a)\chi^\rho(b) - \chi^\rho(a^{-1}b) \overset{L.34}{=} x^2 - y$.

$\cdot\ \chi^\rho(a\,ba\,b^{-1}) = \chi^\rho(ba\,bb^{-1}) = \chi^\rho(ba) = \chi^\rho(ab) = x^2-y$

$\cdot\ \chi^\rho(ab\,ab^{-1}) = \chi^\rho(ab)\chi^\rho(ab^{-1}) - \chi^\rho(b^{-1}a^{-1}ab^{-1}) = (x^2-y)y - \chi^\rho(b^2) =$

$= x^2y - y^2 - (\chi^\rho(b))^2 + \chi^\rho(b^{-1}b) = x^2y - y^2 - x^2 + 2$

So, $x^2 - y = x^2y - y^2 - x^2 + 2$, i.e., $(y-2)(x^2-y-1) = 0$.

Thus, Ch is well defined.

Surjectivity follows from (b) & (c).

(b) $\forall\, V \in \overline{\mathrm{Rep}}_2^{dec}(B_3)\ \exists\, \lambda \in \mathbb{C}^*$ s.t. $V \cong (\mathbb{C}^2, \rho_{(\lambda)})$, $\rho_{(\lambda)}(a) = \rho_{(\lambda)}(b) = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$.

This can be seen by choosing a "nice" basis for V.

On the other hand, any $(\mathbb{C}^2, \rho_{(\lambda)})$ of this type is indeed in $\overline{\mathrm{Rep}}_2^{dec}(B_3)$, and $(\mathbb{C}^2, \rho_{(\lambda)}) \cong (\mathbb{C}^2, \rho_{(\mu)}) \Leftrightarrow \mu = \lambda^{\pm 1}$.

Finally, $\chi^{\rho_{(\lambda)}}(a) = tr\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} = \underbrace{\lambda + \lambda^{-1}}_{\text{takes all values } \in \mathbb{C}}$ and $\chi^{\rho_{(\lambda)}}(ab^{-1}) = tr\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 2$.

Moreover, $\lambda + \lambda^{-1} = \mu + \mu^{-1} \Leftrightarrow \mu = \lambda^{\pm 1}$.

Thus $\overline{\mathrm{Rep}}_2^{rB}(B_3) \xleftrightarrow{1:1} \mathbb{C}/_{\lambda\sim\lambda^{-1}} \xleftrightarrow{1:1} \mathbb{C} \xleftrightarrow{1:1} \{(x,y) \in \mathbb{C}^2 \mid y = 2\}$

$(\mathbb{C}^2, \rho_{(\lambda)}) \longleftarrow\!\shortmid\ \lambda \longmapsto \lambda + \lambda^{-1}$

$x \longleftrightarrow (x, 2)$

$(\mathbb{C}^2, \rho_{(\lambda)}) \longmapsto (\lambda + \lambda^{-1}, 2) = (\chi^{\rho_{(\lambda)}}(a), \chi^{\rho_{(\lambda)}}(ab^{-1}))$

(c) $\det \rho_t(a) = \det\begin{pmatrix}0 & 1 & 0\\ t & 1-t & 0\\ 0 & 0 & 1\end{pmatrix} = -t \Rightarrow \det \rho_t^r(a) = -t$, where $V_t^{rB} = (\mathbb{C}^2, \rho_t^r)$.

Indeed, a basis change transforms ρ_t to $\begin{pmatrix}\rho_t^r & * \\ & * \\ 0\ \ 0 & *\end{pmatrix}$,

where the new basis is $\left(\begin{pmatrix}1\\-t\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-t\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \mathcal{B}$.

and, since $\rho_t(a)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{matrix}0\\0\\1\end{matrix}$, one concludes $\rho_t(a) = M\begin{pmatrix}\rho_t^r(a) & 0\\ & 0\\ 0\ \ 0 & 1\end{pmatrix}M^{-1}$ for some $M \in GL_3(\mathbb{C})$, hence $\det \rho_t^r(a) = \det \rho_t(a)$.

Now, choose an $\alpha \in \mathbb{C}$ with $\alpha^2 = -t$.

$Ch(V_t^{rB} \otimes \theta_\alpha) = (tr(\alpha \rho_t^r(a)), tr(\rho_t^r(ab^{-1}))) = (\alpha(tr\, \rho_t(a) - 1), y) =$

$= (\alpha((2-t)-1), y) = (\alpha(1-t), y)$.

$\alpha(1-t) = \alpha(1+\alpha^{-2}) = \alpha + \alpha^{-1}$ takes all values $\in \mathbb{C}$, since α takes all values $\in \mathbb{C}^*$.

Further, $\rho_t(ab^{-1}) = \begin{pmatrix} 0 & 1 & 0 \\ t & 1-t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t^{-1} & t^{-1} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1-t^{-1} & t^{-1} \\ t & (1-t)(1-t^{-1}) & t^{-1}-1 \\ 0 & 1 & 0 \end{pmatrix}$.

In the basis $\mathcal{B}$, it becomes $\begin{pmatrix} \rho_t^r(ab^{-1}) & \begin{matrix} t^{-1} \\ t^{-1} \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$,

since $\rho_t(ab^{-1})\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t^{-1} \\ t^{-1}-1 \\ 0 \end{pmatrix} = t^{-1}\begin{pmatrix} 1 \\ -t \\ 0 \end{pmatrix} + t^{-1}\begin{pmatrix} 0 \\ 1 \\ -t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

So, $y = \chi^{V_t^{r\mathcal{B}}}(ab^{-1}) = \chi^{V_t^{\mathcal{B}}}(ab^{-1}) - 1 = (1-t)(1-t^{-1}) - 1 = 1 - t - t^{-1} = 1 + \lambda^2 + \lambda^{-2} =$
$= (\lambda + \lambda^{-1})^2 - 1 = x^2 - 1$

(d) Take $V, W \in \overline{Rep_2}(B_3)$ with $Ch(V) = Ch(W)$. We will show, by induction on k, that $\chi^V(a_1^{\varepsilon_1} \dots a_k^{\varepsilon_k}) = \chi^W(a_1^{\varepsilon_1} \dots a_k^{\varepsilon_k})$ for all $a_i \in \{a, b\}$, $\varepsilon_i \in \{1, -1\}$.

- $k=1$: by Lemmas 34 & 36,
  $\chi^W(b^{\pm 1}) = \chi^W(a^{\pm 1})$ and $\chi^V(b^{\pm 1}) = \chi^V(a^{\pm 1})$.
  Since $\chi^W(a) = \chi^V(a)$, we are done.
- $k=2$: $\chi^W(ab^{-1}) = \chi^V(ab^{-1})$ by assumption.
  Again by Lemmas 34 & 36, $\chi^W(ab^{-1}) = \chi^W(b^{-1}a) = \chi^W(a^{-1}b) = \chi^W(ba^{-1})$, and similarly for V. Further, $\chi^W(b^{\pm 2}) = \chi^W(a^{\pm 2}) = \chi^W(a)^2 - \chi^W(a^{-1}a) =$
  $= x^2 - 2$, and similarly for V.
- Suppose the statement is true for $k < s$, where $s \geq 3$, but $\chi^V(a_1^{\varepsilon_1} \dots a_s^{\varepsilon_s}) \neq \chi^W(a_1^{\varepsilon_1} \dots a_s^{\varepsilon_s})$.
  If $a_1 = a_s$ and $\varepsilon_1 = -\varepsilon_s$, then $\chi^V(a_1^{\varepsilon_1} \dots a_s^{\varepsilon_s}) = \chi^V(a_2^{\varepsilon_2} \dots a_{s-1}^{\varepsilon_{s-1}}) \underset{\text{by assumption}}{=} \chi^W(a_2^{\varepsilon_2} \dots a_{s-1}^{\varepsilon_{s-1}}) = \chi^W(a_1^{\varepsilon_1} \dots a_s^{\varepsilon_s})$
  If $a_1 = a_s$ and $\varepsilon_1 = \varepsilon_s$, then $\chi^V(a_1^{\varepsilon_1} \dots a_s^{\varepsilon_s}) = \chi^V(a_1^{\varepsilon_1})\chi^V(a_2^{\varepsilon_2} \dots a_s^{\varepsilon_s}) - \chi^V(a_1^{-\varepsilon_1} a_2^{\varepsilon_2} \dots a_s^{\varepsilon_s}) =$
  $= \chi^V(a)\chi^V(a_2^{\varepsilon_2} \dots a_s^{\varepsilon_s}) - \chi^V(a_2^{\varepsilon_2} \dots a_{s-1}^{\varepsilon_{s-1}})$, and we use the induction assumption.
  If $a_i = a_{i+1}$ for some i, then $\chi^V(a_1^{\varepsilon_1} \dots a_s^{\varepsilon_s}) = \chi^V(a_{i+1}^{\varepsilon_{i+1}} \dots a_s^{\varepsilon_s} a_1^{\varepsilon_1} \dots a_i^{\varepsilon_i})$, and we proceed like in the previous cases.
  We are left with the values $\chi^V(a^{\varepsilon_1} b^{\varepsilon_2} \dots a^{\varepsilon_{s-1}} b^{\varepsilon_s})$, where s is even.
  Since $\chi^V(a^{\varepsilon_1} b^{\varepsilon_2} \dots b^{\varepsilon_s}) + \chi^V(a^{-\varepsilon_1} b^{\varepsilon_2} \dots b^{\varepsilon_s}) = \chi(a^{\varepsilon_1})\chi(b^{\varepsilon_2} \dots b^{\varepsilon_s})$, it suffices to consider the case $\varepsilon_1 = \varepsilon_2$. Further, $aba = bab \Rightarrow aba^{-1} = b^{-1}ab$ & $a^{-1}b^{-1}a = ba^{-1}b^{-1}$ & $a^{-1}b^{-1}a^{-1} = b^{-1}a^{-1}a^{-1}$. So, $a^{\varepsilon_1} b^{\varepsilon_1} a^{\varepsilon_3} = b^{\varepsilon_3} a^{\varepsilon_1} b^{\varepsilon_1}$, so $\chi^V(a^{\varepsilon_1} \dots b^{\varepsilon_s}) = \chi^V(b^{\varepsilon_3} a^{\varepsilon_1} b^{\varepsilon_1} \dots b^{\varepsilon_s})$, and we are in one of the previous cases. $\boxtimes$

Topological applications of the rep. theory of the B_n.

* The first application is obvious:
  Instead of comparing two topological braids B_1 & B_2 (which is extremely difficult!), one can compare the matrices $\rho(B_1)$ & $\rho(B_2)$ for some $(V,\rho) \in \mathrm{Rep}(B_n)$ (which is much easier!). Thus, any B_n-rep. yields an invariant of braids on n strands.

  ⚠ $B_1 \underset{\text{isotopy}}{\sim} B_2 \quad \begin{matrix} \Rightarrow \\ \cancel{\Leftarrow} \end{matrix} \quad \rho(B_1) = \rho(B_2)$

  so this method does not always give you an answer. However, it is very efficient in practice.

* With some more work, one also gets extremely efficient invariants of knots & links:

$$\boxed{\text{Braids} \xrightarrow{\text{closure}} \text{Links}}$$

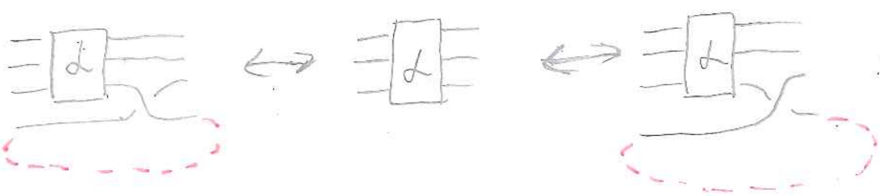
$B =$  $\longmapsto$ $cl(B) :=$ 

(Hopf link)

Alexander's thm (1923): Every link is the closure of some braid.

Markov's thm (1935): $cl(B_1) \sim cl(B_2) \Leftrightarrow B_2$ can be obtained from B_1 by a finite sequence of Markov moves:

M_1 

M_2 

Now, take a family of reps $V_{(n)} \in Rep(B_n)$. It is called "good"

if $\forall \beta \in B_n$, $\chi^{V_{(n)}}(\beta) = \chi^{V_{(n+1)}}(\beta \sigma_n^{\pm 1})$. (We implicitely used the inclusion $B_n \hookrightarrow B_{n+1}$.)

Clearly, $\forall \alpha, \beta \in B_n$, one has $\chi^{V_{(n)}}(\alpha\beta) = \chi^{V_{(n)}}(\beta\alpha)$.

Thus, if for any link L we choose a $\beta \in B_n$ for some n with $cl(\beta) = L$, and we put $p(L) := \chi^{V_{(n)}}(\beta)$, then, by Alexander's & Markov's thms, we get a well defined invariant of links: $p : \text{Links} \to \mathbb{C}$.

It is time for a short excursion into the history of knot theory. Its aim is to classify knots and links, as usual up to isotopy $\sim$ (i.e., up to deformations that do not break the knot).

For instance, the seemingly similar questions

(1)  and  (2)

trefoil knot(s) figure eight knot(s)

have different answers!

For (2), it is "yes". To see this, just play with a piece of rope.

For (1), it is "no". This 19^{th} century problem was solved only in 1985 in the Fields-medal-winning work of Vaughan Jones. He constructed a polynomial-valued invariant

$P : \text{Links} \to \mathbb{Z}[t^{\pm 1}]$ (Jones polynomial of a link)

such that $P(Tr_\ell) \neq P(Tr_r)$.

In fact, Jones wasn't a knot theorist. His polynomial was a by-product of his work on $Rep(B_n)$. More precisely, he constructed a particulary interesting "good" family of reps. Since then, an alternative combinatorial construction of P was found, leading to very efficient computations.

We finish by describing a very rich source of "good" families of B_n-reps — including that of Jones.

Let V be a vector space, and $\sigma \in \mathrm{Aut}_{\mathbb{C}}(V\otimes V)$. The equation

$$\boxed{(\sigma\otimes Id_V)(Id_V\otimes\sigma)(\sigma\otimes Id_V) = (Id_V\otimes\sigma)(\sigma\otimes Id_V)(Id_V\otimes\sigma)} : V^{\otimes 3} \to V^{\otimes 3}$$

(YBE).

(here 

)

is called the Yang-Baxter equation for σ.
It has roots in statistical physics, and since then appeared in a large number of mathematical domains.
Its graphical interpretation above suggests connections to braids, which can be stated very precisely:

Prop. 37: Let V be a vector spaces, and $\sigma \in \mathrm{Aut}_{\mathbb{C}}(V\otimes V)$ be a solution to the YBE. Then

$$\rho_{YBE} = \rho : \boxed{\begin{aligned} B_n &\to \mathrm{Aut}_{\mathbb{C}}(V^{\otimes n}) \\ \sigma_i &\mapsto \underbrace{Id_V\otimes\ldots\otimes Id_V}_{i-1 \text{ times}}\otimes\sigma\otimes\underbrace{Id_V\otimes\ldots\otimes Id_V}_{n-i-1 \text{ times}} \end{aligned}}$$

is a well defined group morphism.

Here, $V^{\otimes n} = \underbrace{(\ldots(V\otimes V)\otimes V\ldots)\otimes V}_{n \text{ times}}$, i.e., $V^{\otimes n} = V^{\otimes(n-1)}\otimes V$, and $V^{\otimes 1} = V$.

In other words, $(V^{\otimes n}, \rho) \in \mathrm{Rep}_n(B_n)$.

□ One has to check that ρ respects the defining relations of B_n:

(1) $\rho(\sigma_i\sigma_j) = Id_V^{\otimes(i-1)}\otimes\sigma\otimes Id_V^{\otimes(j-i-2)}\otimes\sigma\otimes Id_V^{\otimes(n-j-1)} = \rho(\sigma_j\sigma_i)$,
where $j \geq i+2$;

(2) $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1}) \iff$ (YBE).

Finally, $Id_V^{\otimes(i-1)}\otimes\sigma\otimes Id_V^{\otimes(n-i-1)}$ is invertible, since σ is invertible. ⊠

Now, where can we get solutions to the YBE?
From quantum groups! This is a fascinating mathematical notion.
First, because mathematicians do not agree on its definition!
There are some major examples that everyone agrees to call "quantum groups", but it is not clear which set of axioms describes them in the best way.

Second, quantum groups are not groups! They are vector spaces Q with a multiplication $Q \otimes Q \to Q$, comultiplication $Q \to Q \otimes Q$, and some other structure. Like for groups, one can define the reps and tensor products thereof. Even better: they have a so-called R-matrix $R = \sum_i a_i \otimes b_i \in Q \otimes Q$, such that $\forall\, V \in Rep(Q)$,

$$\sigma_R : V \otimes V \to V \otimes V$$
$$v \otimes w \mapsto \sum_i b_i \cdot w \otimes a_i \cdot v$$

is a solution to the YBE. It is, in some sense, the deformation of the trivial solution $v \otimes w \mapsto w \otimes v$.

The major examples of quantum groups are certain "deformations" of Lie groups & Lie algebras. By magic, the YBE solutions σ_R associated to $\forall$ rep. V of such quantum groups give "good" families of reps $(V^{\otimes n}, \rho_{YBE}) \in Rep(B_n)$. The Jones reps are recovered when Q deforms the Lie algebra $sl_2(\mathbb{C})$, corresponding to the Lie group $SL_2(\mathbb{C})$.

Thus, the original construction of the Jones polynomial uses the rep. theory at 2 levels:

1) reps of quantum groups – to construct a nice solution to the YBE;
2) reps of B_n, obtained out of this solution.