Lecture 25: Computing definite integrals

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1 Computing definite integrals

We have seen that definite integrals, and hence areas, can be computed using indefinite integrals:

Theorem 4 (The Fundamental Theorem of Calculus, part 1). If f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b.$$

This means that computation rules for definite integrals follow from those for indefinite integrals.

Certain rules, like linearity, are transported in a straightforward manner. Another example is integration by parts:

Theorem 5 (Integration by parts). Suppose that F and G are antiderivatives of the continuous functions f and g on $[a, b]$. Then

$$\int_a^b [f(x)G(x)]\,dx = F(x)G(x)|_a^b - \int_a^b [F(x)g(x)]\,dx.$$

1 Computing definite integrals

Theorem 5 (Integration by parts). Suppose that F and G are antiderivatives of the continuous functions f and g on $[a, b]$. Then

$$\int_a^b [f(x)G(x)]\,dx = F(x)G(x)\big|_a^b - \int_a^b [F(x)g(x)]\,dx.$$

Example.

$$\begin{aligned}\int_0^1 \arctan x\,dx &= [x\arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\\ &= 1\cdot\frac{\pi}{4} - 0\cdot 0 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1\\ &= \frac{\pi}{4} - \frac{1}{2}(\ln(1+1^2) - \ln(1+0^2)) = \frac{\pi}{4} - \frac{1}{2}\ln(2) \approx 0.44.\end{aligned}$$

Note that $1 + x^2 > 0$ on $[0, 1]$, so that the function $\frac{1}{2}\ln(1+x^2)$ is well defined, and is the primitive of $\frac{x}{1+x^2}$ on $[0, 1]$.

2 Substitution for definite integrals

Substitution for definite integrals is slightly more delicate, since one needs to take care of the integration limits.

There are two ways to deal with it.

Method 1. Use u-substitutions only on the level of indefinite integrals:

$$\int_a^b f(g(x))g'(x)\,dx = [F(g(x))]_a^b,$$

where $F = \int f(x)\,dx$.

Method 2. A somewhat more direct computation is also possible:

Theorem 6 (u-substitution). Suppose that f is continuous on $[a, b]$ and takes values in $[c, d]$, and g and g' are continuous on $[c, d]$. Put $F = \int f(x)\,dx$. Then

$$\int_a^b [f(g(x))g'(x)]\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

2 Substitution for definite integrals

Example 1. Let us use the first method to evaluate $\int_0^2 x(x^2+1)^3\,dx$.

To compute the corresponding indefinite integral, we denote $u = x^2 + 1$, so that $du = 2x\,dx$, and

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C.$$

Therefore,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_0^2 = \left[\frac{(x^2+1)^4}{8}\right]_0^2 = \frac{625}{8} - \frac{1}{8} = 78.$$

2 Substitution for definite integrals

Let us use the 2nd method to evaluate the same integral $\int_0^2 x(x^2+1)^3\,dx$.

That is, we start with the substitution, which is here $u(x) = x^2 + 1$, so that

- ✓ $du = 2x\,dx$;
- ✓ $u(0) = 1$;
- ✓ $u(2) = 5$.

Therefore,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^5 = \frac{1}{2}\left(\frac{5^4}{4} - \frac{1^4}{4}\right) = 78.$$

2 Substitution for definite integrals

Example 2. Let us evaluate $\int_1^3 \frac{\cos(\pi/x)}{x^2}\,dx$.

We put $u(x) = \frac{\pi}{x}$, so that

- ✓ $du = -\frac{\pi}{x^2}\,dx$, in other words, $\frac{1}{x^2}\,dx = -\frac{1}{\pi}\,du$;
- ✓ $u(1) = \pi$;
- ✓ $u(3) = \pi/3$.

Therefore,

$$\int_1^3 \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\int_{\pi}^{\pi/3} \cos u\,du$$

$$= -\frac{1}{\pi}\sin u\Big]_{\pi}^{\pi/3} = -\frac{1}{\pi}(\sin(\pi/3) - \sin\pi) = -\frac{\sqrt{3}}{2\pi} \approx -0.276.$$

2 Substitution for definite integrals

Example 3. Let us evaluate $\int_0^{\pi/4} \sqrt{\tan x}\frac{1}{\cos^2 x}\,dx$.

We put $u = \tan x$, so that

- ✓ $du = \frac{1}{\cos^2 x}\,dx$;
- ✓ $u(0) = 0$;
- ✓ $u(\pi/4) = 1$.

Therefore,

$$\int_0^{\pi/4} \sqrt{\tan x}\frac{1}{\cos^2 x}\,dx = \int_0^1 \sqrt{u}\,du = \left.\frac{u^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}.$$

2 Substitution for definite integrals

Example 4. Let us prove, without evaluating integrals, that

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx.$$

In the second integral, we put $u = \frac{\pi}{2} - x$, so that

- ✓ $du = -dx$;
- ✓ $u(0) = \pi/2, \qquad u(\pi/2) = 0$;
- ✓ $\cos x = \cos(\pi/2 - u) = \sin u$.

Therefore,

$$\int_0^{\pi/2} \cos^n x \, dx = -\int_{\pi/2}^{0} \sin^n u \, du = \int_0^{\pi/2} \sin^n x \, dx.$$

⚠ Here using Method 1, i.e. first computing indefinite integrals $\int \sin^n x \, dx$ and $\int \cos^n x \, dx$, would be much more tedious.

Exercise. Compute $\int \sin^2 x \, dx$ and $\int \sin^3 x \, dx$.

2 Substitution for definite integrals

Example 5. Let us evaluate $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$.

Put $u(x) = \frac{1}{x}$, so that

- ✓ $du = -\frac{1}{x^2}\,dx$, in other words, $du = -u^2\,dx$ and $dx = -\frac{1}{u^2}\,du$;
- ✓ $u(-1) = -1$;
- ✓ $u(1) = 1$;
- ✓ $\frac{1}{1+x^2} = \frac{1}{1+(1/u)^2} = \frac{u^2}{1+u^2}$.

Therefore,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = -\int_{-1}^{1} \frac{u^2}{1+u^2}\frac{1}{u^2}\,du = -\int_{-1}^{1} \frac{1}{1+u^2}\,du,$$

so the integral is equal to its negative and hence equal to zero.

But $\frac{1}{1+x^2} > 0$ on $[-1,1]$ implies $\int_{-1}^{1} \frac{1}{1+x^2}\,dx > 0$.

How is it possible? It happened because $u(x) = \frac{1}{x}$ was not defined on all the interval $[-1,1]$, having a singularity at $x = 0$.

2 Substitution for definite integrals

The right way to compute $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$ is to recognise the derivative of arctan:

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = [\arctan x]_{-1}^{1} = \arctan(1) - \arctan(-1) = \frac{\pi}{4} - (-\frac{\pi}{4}) = \frac{\pi}{2}.$$