Lecture 24: Areas and definite integrals

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1 Areas: reminder

Let us summarise the (very general!) definition of the area under the graph of a "nice" function $f(x)$ on $[a, b]$, seen in the last lecture.

For integers N getting larger and larger, do the following:

1) Divide $[a, b]$ into N equal subintervals of size $\Delta_N = \frac{b-a}{N}$.

2) Choose some points $x_1^*, x_2^*, \ldots, x_N^*$, one in each subinterval.

3) Approximate the area under the curve $y = f(x)$ restricted to the kth subinterval I_k by the area of the rectangle $\Delta_N \times f(x_k^*)$.
(That is, you approximate the values of $f(x)$ on I_k by $f(x_k^*)$.)

4) Compute the total area $\sum_{k=1}^{N} f(x_k^*)\Delta_N$ of these small rectangles.

As N gets larger, these sums turn out to get closer to some number A, which is declared to be the area we are looking for:

$$A = \lim_{N\to+\infty} \sum_{k=1}^{N} f(x_k^*)\Delta_N.$$

1 Areas: reminder

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For familiar plane figures, this definition yields the familiar formulas.

Example. A rectangle of size $b \times c$ can be seen as the plane figure under the graph of $y = c$ on the interval $[0, b]$.
In this case, $f(x_k^*) = c$ for any choice of the points x_k^*, so

$$\sum_{k=1}^{N} f(x_k^*)\Delta_N = \sum_{k=1}^{N} c\Delta_N = c\sum_{k=1}^{N} \Delta_N = cN\frac{b-0}{N} = bc.$$

In particular,

$$A = \lim_{N\to+\infty} \sum_{k=1}^{N} f(x_k^*)\Delta_N == \lim_{N\to+\infty} bc = bc.$$

We thus recover the familiar formula for the area of a rectangle.

2 Integration: Riemann sums

For some purposes, having all subintervals of the same length Δ_N is not optimal. To remedy this, we shall increase the generality of the definition of area even more, and consider arbitrary points $x_1, x_2, \ldots, x_{N-1}$ that divide $[a, b]$ into N subintervals of lengths

$$\Delta_{N;1} = x_1 - a, \; \Delta_{N;2} = x_2 - x_1, \; \ldots, \; \Delta_{N;N} = b - x_{N-1}.$$

To make sure that the sums we compute do approximate the area under the curve, we shall require the **mesh size** $\max_k \Delta_{N;k}$ of our partitions into subintervals to get close to zero as N increases.

Definition. A function f is **(Riemann) integrable** on $[a, b]$ if the limit

$$\int_a^b f(x)\,dx = \lim_{\max_k \Delta_{N;k} \to 0} \sum_{k=1}^{N} f(x_k^*)\Delta_{N;k}$$

exists, and does not depend on the choices of partitions $\Delta_{N;k}$ & points x_k^*. In such case, this limit is called the **definite integral** of $f(x)$ on $[a, b]$.

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Clearly, for an integrable function f, $\displaystyle\int_a^b f(x)\,dx$ is the (net signed) area between the curve $y = f(x)$ on $[a, b]$ and the x-axis.
In particular, if you want to compute an area using Riemann sums, you may choose the partitions $\Delta_{N;k}$ & the points x_k^* which make your computation the easiest possible.

The integration symbols $\int$ and dx can be viewed as the "limit forms" of $\sum$ and $\Delta_{N;k}$ respectively.

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At this point, the definition does not seem particularly helpful, for the following reasons:

1) Whether any non-constant function is integrable at all is a mystery.

2) Even if some functions are, it is absolutely unclear how to compute the scary limit from the definition.

3) Finally, why do we use the same symbol for these strange limits and for antiderivatives? Aren't there enough symbols in maths?

We shall start answering these questions today.

3 Integrability

It is not easy to determine precisely which functions are integrable. The good news is that all “reasonably nice” functions are integrable:

Theorem. For a function f on $[a, b]$ to be integrable, it suffices to be:

- ✓ continuous;
- ✓ or, more generally, piecewise defined, continuous on each piece, and having removable or jump continuities at the gluing points of the pieces.

Before explaining how to compute definite integrals, we shall list their elementary properties.

4 Properties of definite integrals

Definite integrals, like indefinite ones, behave well w.r.t. scalar factors, sums, and differences:

Theorem 1. Suppose that both functions f and g are integrable on $[a, b]$. Let c be a constant. Then the functions cf, $f + g$, and $f - g$ are integrable, and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

More generally, we have

$$\int_a^b (c_1 f_1(x) + \cdots + c_n f_n(x))\,dx = c_1\int_a^b f_1(x)\,dx + \cdots + c_n\int_a^b f_n(x)\,dx.$$

These properties follow from the corresponding properties of sums and limits.

4 Properties of definite integrals

To simplify various formulas, it is beneficial to define the integral symbol $\int_a^b$ beyond the case $a < b$ we have been considering now, as follows:

Definition.

- ✓ If a is in the domain of f, define $\displaystyle\int_a^a f(x)\,dx = 0$.
- ✓ If f is integrable on $[a, b]$, define $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

The former formula is consistent with the intuition of areas: on the interval of zero length, the net signed area must be zero.

The latter formula is a useful convention, although in some cases it has physical meaning: if $f(x)$ is the gravity force between the unit masses placed at the point x and at the origin, then $\int_a^b f(x)\,dx$ is the work required to move the mass from a to b. Of course, moving it in the opposite direction would require negative work on our side, since it will all be done by the gravity!

4 Properties of definite integrals

Theorem 2. Suppose that f is integrable on an interval containing a, b, c. Then, no matter how the points a, b, c are ordered, we have

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

This result is obvious from the area point of view: for $a < b < c$, the total area under the graph on $[a, c]$ is the sum of the areas on $[a, b]$ and on $[b, c]$.

Theorem 3. If f and g are integrable on $[a, b]$, and $f(x) \geqslant g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx.$$

In particular, if f is integrable on $[a, b]$, and $f(x) \geqslant 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geqslant 0$.

These results also have a clear interpretation in terms of areas.

5 The fundamental theorem of calculus

Our scary definition of the area has several advantages:

- ✓ It works for very general plane shapes.
- ✓ It leads to mathematically rigorous proofs of intuitive properties of the area.
- ✓ It can be adapted to define volumes, lengths, masses etc.

However, computing Riemann sums for anything more complicated than a constant function is a nightmare!

In practice, definite integrals (and areas) are evaluated using the following deep result, which is at the heart of calculus, relating differential and integral calculi, or else tangent line and area computation.

5 The fundamental theorem of calculus

Theorem 4 (The Fundamental Theorem of Calculus, part 1). If f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

That is, if you know an antiderivative of f, you can easily compute the definite integral, and hence the area under the graph of f, on any interval!

Example. As an antiderivative of x^2 we can choose $\frac{x^3}{3}$. So, the area under the graph $y = x^2$ on $[0, 1]$ is

$$\int_0^1 x^2\,dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

This coincides with the result obtained using Riemann sums. Even better: we see that the area under the graph $y = x^2$ on any interval $[a, b]$ is $\frac{1}{3}(b^3 - a^3)$.

Later we shall explain where this connexion between integrals and derivatives comes from. For the moment, we shall keep discussing the meaning and variations of the theorem.

5 The fundamental theorem of calculus

Theorem 4 (The Fundamental Theorem of Calculus, part 1). If f is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

There are several conventions to denote $F(b) - F(a)$ more compactly:

$$F(x)\Big]_a^b, \quad \Big[F(x)\Big]_a^b, \quad F(x)\Big|_a^b.$$

So, one can write for example $\int_a^b x^2\,dx = \frac{1}{3}x^3|_a^b$.

Observe that the expression $F(b) - F(a)$ does not depend on the choice of the antiderivative (where we can add an arbitrary constant), since

$$(F(b) + C) - (F(a) + C) = F(b) - F(a).$$

So, one can reformulate the theorem as

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b.$$

5 The fundamental theorem of calculus

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$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b.$$

Note that here

- ✓ the definite integral $\int_a^b f(x)\,dx$ is a number;
- ✓ the indefinite integral $\int f(x)\,dx$ is a family of functions.

⚠ It is important to apply the fundamental theorem of calculus only where it is applicable.

E.g. if $f(x) = \frac{1}{x^2}$, so that $F(x) = -\frac{1}{x}$, then applying our theorem mechanically on $[-1, 1]$ would yield

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big]_{-1}^{1} = -1 - 1 = -2,$$

which is clearly impossible for a positive f!

The problem is that our f is not even defined everywhere on $[-1, 1]$.

5 The fundamental theorem of calculus

Examples. Applying the theorem we just proved, we can evaluate the following integrals:

$$\int_{-2}^{1} (x^2 - 6x + 12)\, dx = \frac{x^3}{3} - 3x^2 + 12x\Big]_{-2}^{1} =$$

$$= \frac{1}{3} - 3 + 12 - \left(-\frac{8}{3} - 12 - 24\right) = 48.$$

$$\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \sin\theta\big]_{-\pi/2}^{\pi/2} = 1 - (-1) = 2.$$

$$\int_{1}^{4} \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\big]_{1}^{4} = 4 - 2 = 2.$$

5 The fundamental theorem of calculus

$$\int_0^{\pi/4} \left(3x^2 + \frac{1}{\cos^2 x}\right) dx = x^3 + \tan x\Big]_0^{\pi/4} = \frac{\pi^3}{64} + 1.$$

$$\int_0^1 |2x-1|\,dx = \int_0^{1/2} |2x-1|\,dx + \int_{1/2}^1 |2x-1|\,dx =$$

$$= \int_0^{1/2} (1-2x)\,dx + \int_{1/2}^1 (2x-1)\,dx = x - x^2\Big]_0^{1/2} + x^2 - x\Big]_{1/2}^1 =$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) - (0-0) + (1-1) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}.$$

5 The fundamental theorem of calculus

There are many applications in which the fundamental theorem of calculus has a clear meaning for real life questions, all following the following general principle:

Integrating the rate of change of $F(x)$ over $[a, b]$ produces the total change of the value of $F(x)$ as x increases from a to b.

- ✓ If $s(t)$ is the position of the particle in rectilinear motion, then $s'(t) = v(t)$ is the instantaneous velocity, and integrating $v(t)$ computes the displacement between the given times.
- ✓ If $P(t)$ is the population (of plants, animals, people) at time t, then $P'(t) = r(t)$ is the rate of change of population, and its integral is the total change of population.
- ✓ If $p(x)$ is the profit generated by producing and selling x units of product, then $p'(x) = m(x)$ is the "marginal profit" (extra profit resulting from producing and selling one extra unit), and its integral from a to b is the change of profit when the production level increases from a to b units.