Lecture 23:
The area of plane figures

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MA1S11A: Calculus with Applications for Scientists

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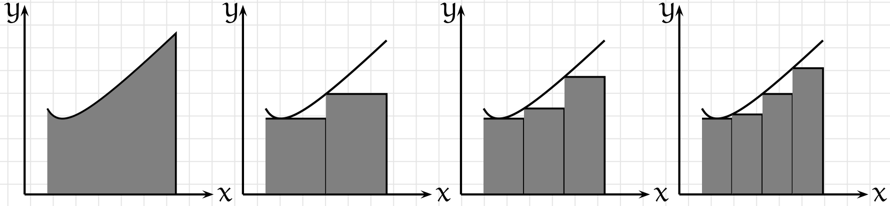
1 How can one compute areas?

Now that we know the basic methods for computing integrals, we will discuss what they can be useful for. The main application, which motivated the whole integral calculus, was computing areas delimited by curves. This clearly boils down to computing areas under curves.

An intuitive and efficient method for computing areas of figures is the **method of exhaustion**, used in Ancient Greece since 5th century BC. It consists in approximation by glued rectangles and triangles, whose area we know how to compute.

For areas under curves, this method looks as follows:



2 Areas: example

Example. Let us compute the area under the curve $y = x^2$ when x varies from 0 to 1.

To approximate this curved shape by union of rectangles, split the segment $[0,1]$ into N segments of length $1/N$:

$$[0, 1/N]\ ,\ [1/N, 2/N]\ ,\ \dots\ ,\ [(N-1)/N, 1].$$

The rectangle over each segment $[k/N, (k+1)/N]$ delimited by the curve $y = x^2$ has width $1/N$ and height $(k/N)^2$ (the value of $f(x) = x^2$ at k/N). This is so because the function $f(x) = x^2$ is increasing on $[0,1]$.

The total area of these rectangles is

$$\begin{aligned}
&\frac{1}{N}\cdot 0^2 + \frac{1}{N}\cdot (1/N)^2 + \frac{1}{N}\cdot (2/N)^2 + \dots + \frac{1}{N}\cdot ((N-1)/N)^2 \\
&= \frac{1}{N^3}(1^2 + 2^2 + \dots + (N-1)^2) \\
&\overset{(*)}{=} \frac{(N-1)N(2N-1)}{6N^3} = \frac{1}{6}\left(1 - \frac{1}{N}\right)\left(2 - \frac{1}{N}\right),
\end{aligned}$$

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$$= \frac{1}{N^3}(1^2 + 2^2 + \cdots + (N-1)^2)$$

$$\overset{(*)}{=} \frac{(N-1)N(2N-1)}{6N^3} = \frac{1}{6}\left(1 - \frac{1}{N}\right)\left(2 - \frac{1}{N}\right),$$

which tends to $\dfrac{2}{6} = \dfrac{1}{3}$ as N increases without bound.

So, it is sensible to define the area under $y = x^2$ on $[0, 1]$ to be $1/3$.

It is intuitively plausible (but not obvious!) that

1) this method will work for any continuous function on an interval;
2) approximation by other families of rectangles yields the same value.

3 Digression: sigma notation

As it should have become apparent by now, to deal with areas, we shall be using long sums with many terms. It is beneficial to get used to the relevant mathematical notation.

Suppose that we have a function $f(x)$, and need to compute the sum

$$f(m) + f(m+1) + \cdots + f(n).$$

In such a case, the symbol $\sum$ (coming from the capital Greek letter Σ (sigma) that usually denotes sums) is used:

$$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + \cdots + f(n).$$

In this formula,

- ✓ k is a summation index, or summation variable;
- ✓ the "$k = m$" below Σ is the starting value of k;
- ✓ "n" above Σ is the ending value of k.

3 Digression: sigma notation

Example. The formula for the sum of squares we used earlier is written as

$$\sum_{k=1}^{N-1} k^2 = \frac{(N-1)N(2N-1)}{6}.$$

Theorem. The following summation formulas are true:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

These formulas are often used in practice, so it is good to remember them.

For $n = N-1$, the second claim yields the formula we used:

$$\sum_{k=1}^{N-1} k^2 = \frac{(N-1)(N-1+1)(2(N-1)+1)}{6} = \frac{(N-1)N(2N-1)}{6}.$$

3 Digression: sigma notation

Scalar factors, sums, and differences behave well with the sigma notation:

$$\sum_{k=m}^{n} cf(k) = c \sum_{k=m}^{n} f(k),$$

$$\sum_{k=m}^{n} (f(k) \pm g(k)) = \sum_{k=m}^{n} f(k) \pm \sum_{k=m}^{n} g(k).$$

Using the basic summation formulas and properties of sums, we can compute more sums, e.g.

$$\sum_{k=1}^{n} (k^2 - 2k) = \sum_{k=1}^{n} k^2 - 2 \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} =$$

$$= \frac{2n^3 + 3n^2 + n - 6n^2 - 6n}{6} = \frac{2n^3 - 3n^2 - 5n}{6}.$$

Exercise. Compute $\sum_{k=1}^{n} (2k-1)^2$, $\sum_{m=3}^{n} m^2(m-2)$.

4 Areas: general case

Let us describe the general recipe for computing the area A under a function $f(x)$ on $[a, b]$. For this to make sense we need to have $f(x) \geqslant 0$ on $[a, b]$.

We will approximate the area by N rectangles of equal width $\Delta_N = \dfrac{b-a}{N}$.

That is, we divide $[a, b]$ into N subintervals of equal length by the points $x_0 = a,\ x_1 = a + \Delta_N,\ x_2 = x_1 + \Delta_N,\ \ldots x_{N-1} = x_{N-2} + \Delta_N,\ x_N = b.$

Let us select arbitrary points $x_1^*, \ldots, x_N^*$ in each of these subintervals.
Each point x_k^* defines a rectangle $\Delta_N \times f(x_k^*)$ which approximates the area under the graph on $[x_{k-1}, x_k]$.
As an approximation to the whole area A, we take

$$f(x_1^*)\Delta_N + \cdots + f(x_N^*)\Delta_N = \sum_{k=1}^{N} f(x_k^*)\Delta_N.$$

Definition. The **area** under the curve $y = f(x)$ on $[a, b]$ is defined as

$$\lim_{N \to +\infty} \sum_{k=1}^{N} f(x_k^*)\Delta_N.$$

4 Areas: general case

Definition. The **area** under the curve $y = f(x)$ on $[a, b]$ is defined as

$$\lim_{N \to +\infty} \sum_{k=1}^{N} f(x_k^*) \Delta_N .$$

There are several things in this definition you should be unhappy about:

✓ We are using the limit $\lim\limits_{N \to +\infty}$, where N is an integer; this is different from the limits $\lim\limits_{x \to +\infty}$ that we considered before. Fortunately, the definitions and the behaviour of these two types of limits are very close, so you can work with $\lim\limits_{N \to +\infty}$ as if it were $\lim\limits_{x \to +\infty}$.

✓ It is true (but not obvious!) that, for a continuous f,

1) the limit in the definition does exist;

2) a different choice of the points x_k^* yields the same value.

The most common choices for the x_k^* are x_{k-1}, x_k, and $\frac{x_{k-1}+x_k}{2}$.

If we do not assume $f(x) \geqslant 0$, we get the definition of the **net signed area** between the curve $y = f(x)$ and the x-axis, i.e., the difference between the area above the curve and the difference below the curve.