Lecture 22: Integration by parts and u-substitution

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MA1S11A: Calculus with Applications for Scientists

December 1, 2017

1 Integration vs differentiation

From our first lecture on integration you might have felt that, even though integrals and derivatives are dual objects, there is an important difference between computing them. It is often formulated as follows:

> Differentiation is mechanics, integration is art.

Indeed, differentiation is algorithmic: if you learned

- ✓ the derivatives of basic functions, and
- ✓ the key rules for gluing them together (product, quotient, chain and inverse function rules),

then you are able to differentiate most functions one meets in sciences.

There are just a few tricks to simplify computations, like logarithmic differentiation.

1 Integration vs differentiation

Differentiation is mechanics, integration is art.

Integration, on the contrary, comes without any general algorithms. We will learn some methods, and in each example it is up to you to choose:

- ✓ the integration method (u-substitution, integration by parts etc.), and
- ✓ auxiliary data for the method (e.g., the base change $u = g(x)$ in u-substitution).

Even worse:

- ✓ different methods might work for the same problem, with different efficiency;
- ✓ the integrals of some elementary functions are not elementary, e.g. $\int e^{-x^2}\,dx$.

You can find more details by clicking here.

The recipe is

- ✓ have enough practice;
- ✓ take each problem as a challenge (like a crosswords puzzle!).

2 Mnemonics for u-substitution

Theorem 3 (u-substitution). Suppose that $F(x)$ is an antiderivative of $f(x)$. Then the function $f(g(x))g'(x)$ is integrable, and

$$\int [f(g(x))g'(x)]\,dx = F(g(x)) + C.$$

To present this rule in a more intuitive form, we need to recall an alternative notation for derivatives: if $u = g(x)$, we write $g'(x) = \frac{du}{dx}$, or else

$$du = g'(x)\,dx.$$

⚠ This is an equation for numbers: du and dx are not numbers, but just placeholders for easy book-keeping.

Now, in Theorem 3, put $u = g(x)$. You can view it as a change of variables $x \rightsquigarrow u$. The integral $\int f(u)\,du$ can then be evaluated in two ways:

- ✓ $\int f(u)\,du = F(u) = F(g(x))$, where F is an antiderivative f;
- ✓ $\int f(u)\,du = \int f(g(x))g'(x)\,dx$.

Theorem 3 simply states that the two methods yield the same result, up to a constant.

3 More examples of u-substitution

Example 1. Let us evaluate the integral $\int \frac{1}{x \ln x}\, dx$.

Putting $u = \ln x$, we get $du = (\ln x)' dx = \frac{dx}{x}$, so

$$\int \frac{1}{x \ln x}\, dx = \int \frac{du}{u} = \ln u + C = \ln(\ln x) + C.$$

⚠ This example illustrates a general principle: it is often reasonable to choose as $u = g(x)$ the "unpleasant part" of the function you are integrating.

3 More examples of u-substitution

Example 2. Let us evaluate the integral $\int x^2\sqrt{x-1}\,dx$.

Here the "unpleasant part" is $\sqrt{x-1}$. Putting $u=\sqrt{x-1}$, we get $u^2=x-1$, so $2u\,du=dx$, and

$$\int x^2\sqrt{x-1}\,dx = \int (u^2+1)^2\cdot u\cdot 2u\,du = \int (u^4+2u^2+1)\cdot 2u^2\,du$$
$$= \int (2u^6+4u^4+2u^2)\,du = \frac{2}{7}u^7+\frac{4}{5}u^5+\frac{2}{3}u^3+C$$
$$= \frac{2}{7}(x-1)^{7/2}+\frac{4}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C.$$

⚠ This example illustrates a useful trick: to get a relation between du and dx, it can be beneficial to first simplify the equality $u=g(x)$.

3 More examples of u-substitution

Example 3. Let us evaluate the integral $\int (\cos x)^3 \, dx$.

Here the "unpleasant part" is $\cos(x)$, but the substitution $u = \cos x$ does not simplify things, since under the integral sign we are lacking $-\sin x = \frac{du}{dx}$. So, we should try something else.

Noticing that $\cos x \, dx = d(\sin x)$, we put $u = \sin x$ and rewrite our integral:

$$\int (\cos x)^2 \, d(\sin x) = \int (1 - (\sin x)^2) \, d(\sin x) = \int (1 - u^2) \, du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}(\sin x)^3 + C.$$

⚠ In general, to integrate trigonometric functions you often need to manipulate them in different ways, for which you need to know basic trigonometric formulas!

4 Integration by parts

From the product rule for derivatives, we deduce the second fundamental integration rule:

Theorem 4 (Integration by parts). Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively. Then

$$\int [f(x)G(x)]\,dx + \int [F(x)g(x)]\,dx = F(x)G(x) + C.$$

Proof. product the chain rule, we obtain

$$(F(x)G(x))' = F'(x)G(x) + F(x)G'(x) = f(x)G(x) + F(x)g(x).$$

□

This rule is usually applied in the following way:

$$\int [f(x)G(x)]\,dx = F(x)G(x) - \int [F(x)g(x)]\,dx.$$

That is, computing the antiderivative of $f(x)G(x)$ reduces to computing the antiderivative of $F(x)g(x)$, which can be substantially simpler.

Mnemonics for integration by parts: $\int G\,dF = FG - \int F\,dG$.

4 Integration by parts

Example 4. Let us evaluate the integral $\int xe^x\,dx$.

The obvious decomposition of xe^x as a product is $x \cdot e^x$.

- ✓ For e^x, integration and differentiation yield the same result e^x.
- ✓ For x, the derivative $x' = 1$ is simpler that the integral $\int x\,dx = \frac{x^2}{2}$.

So, it makes sense to apply integration by parts with $G(x) = x$, $f(x) = e^x$ (in which case $g(x) = 1$, and we can take $F(x) = e^x$). We get

$$\int xe^x\,dx = xe^x - \int 1 \cdot e^x\,dx = xe^x - e^x + C.$$

Note that if we instead apply integration by parts with $f(x) = x$, $G(x) = e^x$ (so that $F(x) = \frac{x^2}{2}$, $g(x) = e^x$), we get

$$\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx,$$

and we need to compute a more complicated integral!

4 Integration by parts

Example 5. Let us evaluate the integral $\int x^2 e^x \, dx$.

Using integration by parts with $G(x) = x^2$, $f(x) = e^x$ (in which case $g(x) = 2x$, and we can take $F(x) = e^x$), we get

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx = x^2 e^x - 2xe^x + 2e^x + C,$$

since we already know $\int xe^x \, dx$.

The same methods works for any polynomial P of degree n, and yields

$$\int P(x)e^x \, dx = \big(P(x) - P'(x) + P''(x) - \cdots + (-1)^n P^{(n)}(x)\big)e^x + C.$$

4 Integration by parts

Example 6. Let us evaluate $\int \ln x \, dx$.

Since the function $\ln x$ is very pleasant to differentiate ($\ln' x = \frac{1}{x}$), we could try to choose it as one of the factors. The second factor then has to be the constant function 1.

Using integration by parts with $G(x) = \ln x$, $f(x) = 1$ (in which case $g(x) = \frac{1}{x}$, and we can take $F(x) = x$), we get

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + C.$$

⚠ Trivial factorisations like $h(x) = 1 \cdot h(x)$ and artificial factorisations like $h(x) = x \cdot \frac{h(x)}{x}$ are sometimes useful in integration by parts.

4 Integration by parts

Example 7. Let us evaluate $\int e^x \sin x \, dx$.

Using integration by parts with $G(x) = \sin x$, $f(x) = e^x$ (so, $g(x) = \cos x$, and we take $F(x) = e^x$), we get

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

Now comes the tricky part: let us integrate by parts again, with $G(x) = \cos x$, $f(x) = e^x$ (so, $g(x) = -\sin x$, and we take $F(x) = e^x$):

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx.$$

Summarising, we get

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx,$$

hence

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

Exercise. Compute $\int e^x \cos x \, dx$.

5 Choosing between the two methods

We will now look at examples where no integration method is imposed.

Example 8. Compute the integral $\int \sin(\ln x)\,dx$.

Approach 1. Let us try to apply a u-substitutions.
In a composition, the "unpleasant part" is often the internal function. Here it is $\ln x$.
Putting $u = \ln x$, so that $e^u = x$, and $e^u\,du = dx$, we get

$$\int \sin(\ln x)\,dx = \int \sin(u)\,e^u\,du.$$

In Example 7, we computed the latter integral using integration by parts:

$$\int e^u \sin u\,du = \frac{1}{2}e^u(\sin u - \cos u) + C,$$

so

$$\int \sin(\ln x)\,dx = \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + C.$$

Approach 2. Start with an integration by parts. This is left for you as an exercise.

5 Choosing between the two methods

Example 9. Compute the integral $\int x^3 e^{x^2}\,dx$.