Lecture 21: Antiderivatives. Definition and computation

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MA1S11A: Calculus with Applications for Scientists

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1 The notion of antiderivative

Today we are starting the last chapter of our course: Integral calculus.

We shall first learn what are its basic objects, and how to deal with them. Later we shall see why they are useful in science.

It is often important for applications to solve the problem reverse to differentiation: find a function if we know only its derivative.

Example. You see the instantaneous velocity on the speedometer all the time, and wonder how far you progressed over the given period of time. Mathematically, from $x'(t)$ you want to deduce $x(t_1) - x(t_2)$.

Definition. A function F is called an **antiderivative** of a function f if

$$\frac{dF(x)}{dx} = f(x).$$

If such an F exists, the function f is called **integrable**.

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Recall a fact we have learned about derivatives:

Theorem. $f'(x) = g'(x)$ on $(a, b) \iff f(x) = g(x) + c$ on (a, b) for some $c \in \mathbb{R}$.

It can be reformulated in terms of antiderivatives:

Theorem 1. F_1 and F_2 are two antiderivatives of f on $(a, b) \iff$ $F_1(x) = F_2(x) + C$ on (a, b) for some $C \in \mathbb{R}$.

In words, the antiderivative of a function on an interval, if exists, is well defined up to a constant.

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Example. We have computed the following derivative:

$$(\ln|x|)' = \frac{1}{x} \text{ on } (-\infty, 0) \sqcup (0, +\infty).$$

It means that $\ln|x|$ is an antiderivative of $\frac{1}{x}$. But then the function

$$F(x) = \begin{cases} \ln|x| + \sqrt[3]{\pi}, & x < 0, \\ \ln|x| - e, & x > 0 \end{cases}$$

is another antiderivative of $\frac{1}{x}$.

1 The notion of antiderivative

There is an alternative "integral" notation for saying that F is an antiderivatives of f:

$$\int f(x)\,dx = F(x) + C.$$

The expression $\int f(x)\,dx$ is called the **indefinite integral**.

By definition, $(\int f(x)\,dx)' = f(x)$.

⚠ The constants C should be handled with care. For instance,

$$\int x\,dx = \frac{x^2}{2} + C \quad \Longrightarrow \quad \int\int x\,dx\,dx = \frac{x^3}{6} + Cx + D.$$

Also, you should leave the arbitrary constant till the very end of the computation, to avoid something like this:

$$x^2 + C = \int 2x\,dx = 2\int x\,dx = 2\left(\frac{x^2}{2} + C\right) = x^2 + 2C,$$

which yields $C = 2C$, so $C = 0$.

2 Computing antiderivatives

In simplest situations, antiderivatives are computed by recognising derivatives, since integration and differentiation are inverse operations:

differentiation formula	integration formula
$(x^r)' = rx^{r-1}$	$\int rx^{r-1}\,dx = x^r + C$
$\sin' x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\cos' x = -\sin x$	$\int -\sin x\,dx = \cos x + C$
$\tan' x = \frac{1}{\cos^2 x}$	$\int \frac{dx}{\cos^2 x} = \tan x + C$
$\cot' x = -\frac{1}{\sin^2 x}$	$\int -\frac{dx}{\sin^2 x} = \cot x + C$
$\arcsin' x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
$\arctan' x = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \arctan x + C$
$(\log_a x)' = \frac{1}{x \ln a}$	$\int \frac{dx}{x \ln a} = \log_a x + C$
$(a^x)' = a^x \ln a$	$\int a^x \ln a\,dx = a^x + C$

For this reason, it is important to learn the derivatives of basic functions.

2 Computing antiderivatives

For not-in-the-differentiation-table functions, integration is far from obvious. What is, say, the antiderivative of $f(x) = \frac{1}{\sin(x)}$?

According to the brick-and-mortar approach, in order to learn how to integrate, it is crucial to understand relations between integration and gluing operations for functions. The following results are translations of the corresponding properties of derivatives.

Theorem 2. Suppose that the functions $f(x)$ and $g(x)$ are integrable, and $c \in \mathbb{R}$ is a constant. Then

- ✓ $\int cf(x)\,dx = c\int f(x)\,dx$,
- ✓ $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$,
- ✓ $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$.

The statements can be written in a more explicit form, e.g.,

$$\int f(x)\,dx = F(x) + C \quad \Longrightarrow \quad \int cf(x)\,dx = cF(x) + C.$$

3 u-substitution

Further, from the chain rule for derivatives, we deduce the following important integration rule:

Theorem 3 (u-substitution). Suppose that $F(x)$ is an antiderivative of $f(x)$. Then the function $f(g(x))g'(x)$ is integrable, and

$$\int [f(g(x))g'(x)]\,dx = F(g(x)) + C.$$

Proof. Applying the chain rule, we obtain

$$(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x).$$

$\square$

Example 1. Let us evaluate the integral

$$\int (x^2+1)^{50} \cdot 2x\,dx.$$

Let $g(x) = x^2 + 1$. Noticing that $(x^2+1)' = 2x$, we can write the integral as

$$\int (x^2+1)^{50} \cdot 2x\,dx = \int g(x)^{50} g'(x)\,dx = \frac{g(x)^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C.$$

3 u-substitution

Example 2. Let us evaluate the integral $\int \frac{\cos x}{\sin^2 x}\,dx$.

Denoting $g(x) = \sin x$, we get

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{g(x)^2} g'(x)\,dx = -\frac{1}{g(x)} + C = -\frac{1}{\sin x} + C.$$

⚠ Always verify your computations by differentiation:

$$\left(-\frac{1}{\sin x}\right)' = -\left(-\frac{1}{\sin^2 x}\right) \cdot \sin' x = \frac{1}{\sin^2 x} \cdot \cos x = \frac{\cos x}{\sin^2 x}.$$

Example 3. Let us evaluate the integral $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx$.

Denoting $g(x) = \sqrt{x}$, we get

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx = \int \cos g(x) \cdot 2g'(x)\,dx = 2 \sin g(x) + C = 2 \sin \sqrt{x} + C.$$