# Lecture 20: Exponential and logarithmic functions

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# 1 More functions

All functions we have seen so far were

- ✓ rational functions;
- ✓ trigonometric functions and their inverses;
- ✓ different combinations thereof.

Today we will extend our kit of basic functions by three more classes, widely used in sciences:

- ✓ irrational power functions;
- ✓ exponential functions;
- ✓ logarithmic functions.

They are widely used in science, e.g., to describe:

- ✓ population growth and spread of disease;
- ✓ the magnitude of an earthquake at different distances from the epicentre;
- ✓ perceived loudness of a sound.

## 2 Power functions

Recall that one defines the expression $a^p$ with rational $p$ step by step:

1) For a natural number $p = n$, put $a^n = a \times a \times \cdots \times a$ ($n$ times).

2) For $p = \frac{1}{n}$, use the fact that the $n$th root function $g(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is inverse to the $n$th power function $f(x) = x^n$.

3) When $p = \frac{m}{n}$ is a positive rational number in its reduced form (i.e., $m$ and $n$ do not have common divisors), put $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

4) When $p = -\frac{m}{n}$ is a negative rational number in its reduced form, put $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$.

5) Finally, declare $a^0 = 1$.

For simplicity, we assume $a > 0$ everywhere above, although some of the power functions are defined for all $a$, or all $a \neq 0$, or all $a \geqslant 0$.

## 2 Power functions

Now, let us define $a^p$ for irrational $p$.

First, find a sequence $p_1, p_2, \ldots$ of rational numbers approximating $p$.
For instance, for $p = \pi$ you may take

$$p_1 = 3, \quad p_2 = 3.1, \quad p_3 = 3.14, \quad p_4 = 3.141, \ldots$$

Then, compute $a^{p_1}, a^{p_2}, \ldots$
These numbers happen to get successively closer to some number (independent of the chosen approximating sequence for $p$), called $a^p$.

## 2 Power functions

Irrational power functions $f(x) = x^p$ inherit most properties of rational power functions:

**Theorem.** For any $a, b > 0$ and $p, q \in \mathbb{R}$, one has:

- ✓ $a^{p+q} = a^p a^q, \qquad a^{p-q} = \frac{a^p}{a^q}, \qquad a^{-q} = \frac{1}{a^q};$
- ✓ $a^{pq} = (a^p)^q;$
- ✓ $(ab)^p = a^p b^p, \qquad \left(\frac{a}{b}\right)^p = \left(\frac{a^p}{b^p}\right);$
- ✓ $a^p \begin{cases} > 1 & \text{if } a > 1 \ \& \ p > 0 \text{ or } a < 1 \ \& \ p < 0, \\ < 1 & \text{if } a > 1 \ \& \ p < 0 \text{ or } a < 1 \ \& \ p > 0 \\ = 1 & \text{if } a = 1 \text{ or } p = 0. \end{cases}$

Moreover, the function $f(x) = x^p$ is differentiable on $(0, +\infty)$ infinitely many times, and

$$\begin{aligned} (x^p)' &= px^{p-1}, \\ (x^p)'' &= p(p-1)x^{p-2}, \\ &\dots \\ (x^p)^{(n)} &= p(p-1)\cdots(p+1-n)x^{p-n}. \end{aligned}$$

## 3 Exponential functions

To get a power function, we fixed the power $p$ in $a^p$ and let $a$ vary. If instead we fix $a > 0$ and let $p$ vary from $-\infty$ to $+\infty$, we get an **exponential function** $f(x) = a^x$.

From the properties of powers, we can deduce properties of exponential functions:

**Theorem.** Given $a > 0$, the exponential function $f(x) = a^x$ satisfies the following properties:

- ✓ $f(x + y) = f(x)f(y)$ (i.e., $f$ transforms sums into products);
- ✓ $f$ is
  - increasing form $\lim_{x \to -\infty} a^x = 0$ to $\lim_{x \to +\infty} a^x = +\infty$ if $a > 1$;
  - decreasing form $\lim_{x \to -\infty} a^x = +\infty$ to $\lim_{x \to +\infty} a^x = 0$ if $a < 1$;
  - constant if $a = 1$: $1^x = 1$;
- ✓ $f$ is continuous.

In particular, if $a \neq 1$, then the function $a^x$ has a horizontal asymptote $x = 0$.

# 3 Exponential functions

Here are the graphs of exponential functions $f(x) = a^x$ in three cases:

$$a > 1, \qquad a = 1, \qquad a < 1.$$


For $a < 1$ we have $a^x = (a^{-1})^{-x}$ and $a^{-1} > 1$, so the graph of $f(x) = a^x$ is obtained from that of $f(x) = (a^{-1})^x$ by symmetry about the $y$-axis. Thus in what follows we will mostly consider the case $a > 1$.

# 4 Logarithmic functions

We have seen that for $a > 1$ the function $f(x) = a^x$ increases from 0 to $+\infty$. As a result it has an inverse, called the **logarithmic function with base** $a$, $f(x) = \log_a x$. By definition,

$$y = \log_a x \qquad \Longleftrightarrow \qquad a^y = x.$$

*Examples.* $\log_{10} 1000 = 3$, $\log_{10} 0.01 = -2$, $\log_2 1024 = 10$,

$$\log_a a = 1, \log_a 1 = 0.$$

The properties of $\log_a$ follow from that of exponential functions:

**Theorem.** Given $a > 1$, the logarithmic function $f(x) = \log_a(x)$ satisfies the following properties:

- ✓ the domain of $f$ is $(0, +\infty)$;
- ✓ $a^{\log_a(x)} = x, \qquad \log_a(a^x) = x$;
- ✓ $\log_a(xy) = \log_a(x) + \log_a(y)$ ($\log_a$ turns products into sums);
- ✓ $\log_a(x/y) = \log_a(x) - \log_a(y), \qquad \log_a(1/y) = -\log_a(y)$;
- ✓ $\log_a(x^p) = p\log_a(x)$ and $\log_a(\sqrt[p]{x}) = \frac{\log_a(x)}{p}$ for any $p \in \mathbb{R}$;

## 4 Logarithmic functions

**Theorem.** Given $a > 1$, the logarithmic function $f(x) = \log_a(x)$ satisfies the following properties:

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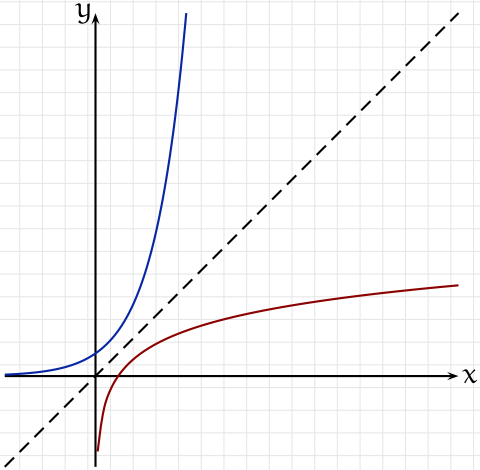
- ✓ $\log_a(xy) = \log_a(x) + \log_a(y)$ ($\log_a$ turns products into sums);
- ✓ $\log_a(x/y) = \log_a(x) - \log_a(y), \qquad \log_a(1/y) = -\log_a(y)$;
- ✓ $\log_a(x^p) = p\log_a(x)$ and $\log_a(\sqrt[p]{x}) = \frac{\log_a(x)}{p}$ for any $p \in \mathbb{R}$;
- ✓ $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$ for any $a, c > 1$, $b > 0$ (base change rule);
- ✓ f is increasing from $\lim_{x\to 0^+} \log_a(x) = -\infty$ to $\lim_{x\to+\infty} \log_a(x) = +\infty$;
- ✓ f is continuous.

⚠ $\log_a(x + y) \neq \log_a(x) + \log_a(y)$.

The function $\log_a$ is also defined for $0 < a < 1$. Its properties are analogous to what we established in the case $a > 1$, with the word *decreasing* replacing *increasing*.

# 4 Logarithmic functions

The graph of $\log_a(x)$ is obtained from that of $a^x$ by applying symmetry about the line $y = x$:

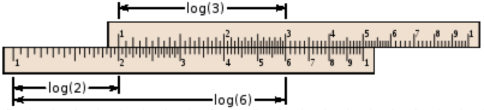


## 5 History of logarithms

Logarithms were introduced by John Napier in the early 17th century to replace tedious multiplication with (much simpler!) addition, combined with the use of slide rules or table look-ups for the values of log:

$$M \cdot N = 10^{\log M} \cdot 10^{\log N} = 10^{\log M + \log N}.$$

Here $\log = \log_{10}$.



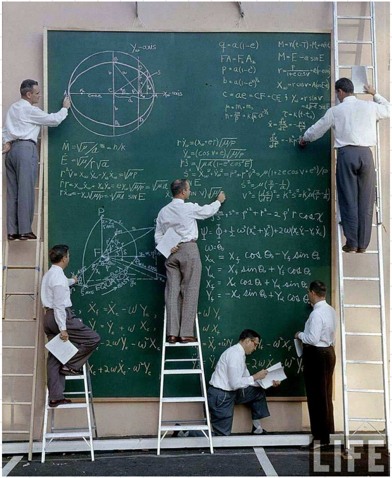

Logarithms are equally useful for extracting roots:

$$\sqrt[n]{x} = 10^{\log(x^{\frac{1}{n}})} = 10^{\frac{1}{n}\log(x)}.$$

Altogether, logarithms proved to be indispensable for fast computations, and tables of logarithms used to be one of the main books that scientists and engineers would use in everyday life before computers and advanced calculators arrived.

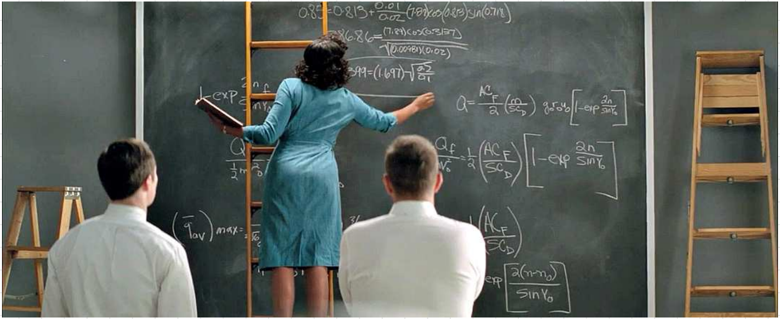
# 5 History of logarithms

This story is not as ancient as you might think: logarithmic tables were extensively used by scientists as late as in the 1960's.




NASA scientists with their board of calculations, 1961, *Life magazine.*

# 5 History of logarithms



A screenshot from *Hidden Figures*.

# 5 History of logarithms

Another advantage of logarithms is that they reduce wide-ranging quantities to smaller scopes. The most famous logarithmic measures in science are:

- ✓ dB (decibel);
- ✓ pH (potential of hydrogen);
- ✓ Richter scale.

## 6 Euler's number

Even though it is not obvious from its definition, the exponential function $f(x) = a^x$ is differentiable on $\mathbb{R}$ as many times as you wish.
Let us try to compute its derivative:

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = f(x)f'(0).$$

Thus, $f'(x)$ is proportional to $f(x)$, with proportionality coefficient $f'(0) = \lim_{h\to 0} \frac{a^h-1}{h}$.

It is intuitively plausible that, when $a$ changes from 1 to $+\infty$, $f'(0)$ continuously changes from 0 to $+\infty$.
So, by the Intermediate Value Theorem, it takes the value 1 at some $a > 0$.
This $a$ turns out to be the **Euler's number** $e = 2.7182818284590\ldots$
We will use the classical notations

$$\exp(x) = e^x, \qquad \ln(x) = \log_e(x).$$

These functions are called the **natural exponential function** and the **natural logarithm**.

# 6 Euler's number

The number $e$ is irrational.

It can be defined in a number of ways:

1) As the only real number satisfying $(e^x)' = e^x$, or, equivalently, $(e^x)'|_{x=0} = 1$.

2) $e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$.

3) $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

4) As the only real number satisfying $\int_1^e \frac{1}{t} \, dt = 1$.

And many more!

## 6 Euler's number

Let us show that the following definitions coincide:.

1) $e$ is the only real number satisfying $(e^x)'|_{x=0} = 1$.

2) $e = \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$.

Denoting $e^h - 1 = t$, we see that $h = \ln(1+t)$, so

$$\frac{e^h - 1}{h} = \frac{t}{\ln(1+t)} = \frac{1}{\frac{1}{t}\ln(1+t)} = \frac{1}{\ln(1+t)^{\frac{1}{t}}}.$$

Since exponential and logarithmic functions are continuous, we conclude

$$(e^x)'|_{x=0} = \lim_{h\to 0}\frac{e^h - 1}{h} = \lim_{t\to 0}\frac{1}{\ln(1+t)^{\frac{1}{t}}} = \frac{1}{\ln(\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}})}.$$

This equals 1 iff $e = \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$.

# 6 Euler's number

Number of known decimal digits of $e$:

| Date | Decimal digits | Computation performed by |
|---|---|---|
| 1690 | 1 | Jacob Bernoulli |
| 1714 | 13 | Roger Cotes |
| 1748 | 23 | Leonhard Euler |
| 1853 | 137 | William Shanks |
| 1871 | 205 | William Shanks |
| 1884 | 346 | J. Marcus Boorman |
| 1949 | 2,010 | John von Neumann (on the ENIAC) |
| 1961 | 100,265 | Daniel Shanks and John Wrench |
| 1978 | 116,000 | Steve Wozniak on the Apple II |

# 7 Derivatives of $a^x$ and $\log_a x$

**Theorem.** Given any $a > 0$, $a \neq 1$, the functions $f(x) = a^x$ and $g(x) = \log_a(x)$ are differentiable everywhere. We have

$$(a^x)' = a^x \ln a, \qquad (\log_a x)' = \frac{1}{x \ln a}.$$

In particular,

$$(e^x)' = e^x, \qquad (\ln x)' = \frac{1}{x}.$$

*Proof.* We have seen that $(e^x)' = e^x$. Using the chain rule, we then deduce

$$(a^x)' = ((e^{\ln a})^x)' = (e^{\ln(a)x})' = e^{\ln(a)x} \cdot (\ln(a)x)' = a^x \ln a.$$

The differentiation rule for inverse functions now yields

$$(\log_a x)' = \frac{1}{(a^y)'|_{y=\log_a x}} = \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \ln a}.$$

□