19-20. Induction & restriction

Question: Given a subgroup $H < G$ of a finite group G, how close are $\mathrm{Rep}\,H$ & $\mathrm{Rep}\,G$?

We have already seen how to go in one direction:

① restriction: $\mathrm{Res}^G_H : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$
$$(V, \rho) \mapsto (V, \rho \circ \iota),$$
where $\iota : H \to G$ is the inclusion map.

Today we will learn how to go in the opposite direction:

② induction: $\mathrm{Ind}^G_H : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$.

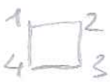
Induction is a powerful tool for constructing new interesting reps. We'll see this on examples. More generally, "many" reps of G can be induced from that of its cyclic subgroups — which we understand very well! Here the word "many" has a precise mathematical meaning; see Artin's theorem.

Ex.: • $S_{n-1} < S_n$, index $= \frac{\# S_n}{\# S_{n-1}} = \frac{n!}{(n-1)!} = n$.

$\mathrm{Res}^{S_n}_{S_{n-1}}(V_\lambda) \overset{\text{Thm 12}}{=} \bigoplus_{\mu \lessdot \lambda} V_\mu$.

Exo: give a fast construct^{n} of the char. table of S_{n-1} out of that of S_n.

• Given $g \in G$ of order d, $\{1, g, g^2, \dots, g^{d-1}\}$ is a subgroup of G, isomorphic to the cyclic group $C_d = \langle t \mid t^d = 1 \rangle$

• $D_8 < S_4$, index $= \frac{\# S_4}{\# D_8} = \frac{4!}{8} = 3$.
$D_8 =$ symmetries of ; the action on the vertices determines the inclusion into S_4 (cf. HW2).

• Main example: $A_n < S_n$, the alternating group, index: 2. You have explored $\mathrm{Res}^{S_4}_{A_4}$ in Tutorial 2. The case of general n will be treated here.

We (finally!) turn to the construction of Ind_H^G.

- Denote $r = \frac{\#G}{\#H}$ the index of H in G.
- Fix coset representatives g_i for G/H, with $g_1 = 1$, and consider the coset decomposition

$$G = g_1 H \sqcup g_2 H \sqcup \ldots \sqcup g_r H.$$

Def.: $\mathrm{Ind}_H^G : \mathrm{Rep}\, H \to \mathrm{Rep}\, G$

$$(V, \cdot) \mapsto \left(\bigoplus_{i=1}^{r} V_i, \tilde{\cdot}\right),$$

where
- all $V_i = V$ as vector spaces; the element in V_i corresponding to $v \in V$ is denoted by v_i
- $\forall g \in G, \forall i, \forall v \in V,\quad g \tilde{\cdot} v_i = (h \cdot v)_j,$ where $h \in H$ and $j \in \{1, 2, .., r\}$ are defined by $g\, g_i = g_j h$ in G.

Prop. 27: (1) The construction $\left(\bigoplus_{i=1}^{r} V_i, \tilde{\cdot}\right)$ above is indeed a G-rep.

(2) Up to G-rep. isomorphism, it does not depend on the choise of coset representatives g_i.

(3) $g_i \tilde{\cdot} V_1 = V_i$, and the restriction $\rho(g_i)|_{V_1} : V_1 \to V_i$ is the identity Id_V.

Thus the map Ind_H^G is well defined. The last point motivates the convenient notation $\mathrm{Ind}_H^G(V) = \bigoplus_{i=1}^{r} g_i V$, common in the literature.

The proof of this and subsequent propositions is straightforward and will be omitted here. The Reader is advised to go through the details, in order to get accustomed to the Ind construction.

Ex.: • $\mathrm{Ind}_{\{1\}}^G (V^{tr}) \cong V^{reg}$

coset decomposit^n: $G = \bigsqcup_{g \in G} g\{1\}$

$V^{tr} \cong \mathbb{C}e$, $\mathrm{Ind}(V^{tr}) = \bigoplus_{g \in G} (\mathbb{C}e)_g = \bigoplus_{g \in G} \mathbb{C}e_g$, $\quad g \cdot e_{g'} = e_{gg'}$ since $gg' = \underbrace{(gg')}_{\text{a coset representative}} \underbrace{1}_{\in H}$

• More generally, $\mathrm{Ind}_H^G (V_H^{reg}) \cong V_G^{reg}$.

- $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}}(V) \cong V$
- $\mathrm{Ind}_{S_{n-1}}^{S_n}(V^{tr}) \cong V^{perm}$

coset decomposit^n: $S_n = \coprod_{i=1}^{n} \tau_i S_{n-1}$, where $\tau_n = \mathrm{Id}$, and $\tau_i = \underbrace{(i\ n)}_{\text{transposit}^n}$ for $i \geq 2$

To see this, it suffices to show $\tau_i S_{n-1} \cap \tau_j S_{n-1} = \{0\}$ for $i \neq j$.
Indeed, $\forall \sigma \in \tau_i S_{n-1}$, $\sigma(n) = i$, $\forall \sigma' \in \tau_j S_{n-1}$, $\sigma(n) = j$, so $\sigma \neq \sigma'$.
Now, $\forall \sigma \in S_n$, $\sigma \tau_i(n) = \sigma(i)$, so $\sigma\tau_i \in \tau_{\sigma(i)} S_{n-1}$.
Then $\mathrm{Ind}(V^{tr}) = \bigoplus_{i=1}^{n} \mathbb{C}e_i$, with $\sigma \cdot e_i = (\underset{m_{S_{n-1}}}{\mu} \cdot e)_{\sigma(i)} = e_{\sigma(i)}$, since S_{n-1} acts trivially on V^{tr}. One recognises V^{perm}!

We now describe some general properties of Ind & Res.

Prop. 28: $\forall V \in \mathrm{Rep}\, H$, $\forall W \in \mathrm{Rep}\, \mathcal{G}$,

(1) Res Ind (V) contains V as a sub-H-rep.
(2) Ind Res$(W) = W \otimes P$, where P is the perm. rep. associated to the (obvious) $\mathcal{G}$-action on the set of cosets $\mathcal{G}/H$

so, Ind: $\mathrm{Rep}\, H \rightleftarrows \mathrm{Rep}\, \mathcal{G}$: Res are close to being mutual inverses

(3) transitivity:

$\mathrm{Ind}_{\mathcal{G}}^{K}\, \mathrm{Ind}_{H}^{\mathcal{G}} = \mathrm{Ind}_{H}^{K}$, $\quad \mathrm{Res}_{L}^{H}\, \mathrm{Res}_{H}^{\mathcal{G}} = \mathrm{Res}_{L}^{\mathcal{G}}$

(4) Ind & Res are monoid morphisms $(\mathrm{Rep}\, H, \oplus, \{0\}) \rightleftarrows (\mathrm{Rep}\, \mathcal{G}, \oplus, \{0\})$

(5) $\mathrm{Hom}_H(V, \mathrm{Res}\, W) \cong \mathrm{Hom}_{\mathcal{G}}(\mathrm{Ind}\, V, W)$
$\mathrm{Hom}_H(\mathrm{Res}\, W, V) \cong \mathrm{Hom}_{\mathcal{G}}(W, \mathrm{Ind}\, V)$

In the category theory language: functors Ind & Res form an adjoint pair

(6) $\chi^{\mathrm{Ind}\, V}(g) = \underset{\substack{i=1,\dots,r \\ g_i^{-1} g g_i \in H}}{\sum} \chi^V(g_i^{-1} g\, g_i)$

(7) **Frobenius reciprocity:** $(\chi^V, \chi^{\mathrm{Res}\, W})_H = (\chi^{\mathrm{Ind}\, V}, \chi^W)_{\mathcal{G}}$.

In practice, Frobenius reciprocity is a particularly important property: it allows one to compute the multiplicity of $W \in \operatorname{Irrep} G$ in $\operatorname{Ind} V$ working entirely in $\operatorname{Rep} H$, which is much simpler than $\operatorname{Rep} G$!

Frobenius reciprocity can be deduced

- either from (5), by restricting yourself to irreducible V & W (using (4)), and then applying Schur's lemma;
- or from (6):

$$(\chi^{\operatorname{Ind} V}, \chi^W)_G = \frac{1}{\# G} \sum_{g \in G} \chi^{\operatorname{Ind} V}(g) \overline{\chi^W(g)} = \frac{1}{\# G} \sum_{g \in G} \sum_{i=1,\dots,r} \chi^V(g_i^{-1} g g_i) \underbrace{\overline{\chi^W(g)}}$$

$$= \frac{1}{\# G} \sum_{h \in H} \#\{g \in G, i \in \{1,\dots,r\} \mid \underset{\Updownarrow}{g_i^{-1} g g_i = h}\} \chi^V(h) \overline{\chi^{\operatorname{Res} W}(h)}$$

$g_i^{-1} g g_i \in H$

$g = g_i h g_i^{-1}$

$$= \frac{1}{\# G} \sum_{h \in H} \underbrace{\#\{i \in \{1,\dots,r\}\}}_{=r} \chi^V(h) \overline{\chi^{\operatorname{Res}(W)}(h)}$$

$= \overline{\chi^W(g_i^{-1} g g_i)} = \overline{\chi^{\operatorname{Res} W}(g_i^{-1} g g_i)}$, since $g_i^{-1} g g_i \in H$

$$= \frac{1}{\# G} \cdot \frac{\# G}{\# H} \sum_{h \in H} \chi^V(h) \overline{\chi^{\operatorname{Res}(W)}(h)} = (\chi^V, \chi^{\operatorname{Res}(W)})_H.$$

Ex.: $\operatorname{Ind}_{S_2}^{S_3} V_2^{sgn} \cong ?$

S_2	Id	(12)
V_2^{tr}	1	1
V_2^{sgn}	1	-1

S_3	Id	(12)	(123)
V_3^{tr}	1	1	1
V_3^{sgn}	1	-1	1
V_3^{st}	2	0	-1

$\operatorname{Res} V_3^{tr} \cong V_2^{tr}$

$\operatorname{Res} V_3^{sgn} \cong V_2^{sgn}$

$\operatorname{Res} V_3^{st} \cong V_2^{tr} \oplus V_2^{sgn}$

$\forall W \in \operatorname{Irrep} S_3$, its multiplicity in $\operatorname{Ind} V_2^{sgn}$ is

$$(\chi^{\operatorname{Ind} V_2^{sgn}}, \chi^W)_{S_3} = (\chi^{V_2^{sgn}}, \chi^{\operatorname{Res} W})_{S_2} = \begin{cases} 1, & W = V_3^{sgn} \text{ or } V_3^{st} \\ 0, & W = V_3^{tr}. \end{cases}$$

So, $\operatorname{Ind}_{S_2}^{S_3} V_2^{sgn} \cong V_3^{sgn} \oplus V_3^{st}$.

Res & Ind behave particularly well when H is of index $r=2$, which is the case in our target example $A_n < S_n$.

So, from now on, we suppose $\underline{r=2}$. In part., $H<G$ is a normal subgroup.

In this case, there are interesting involutions both on $\mathrm{Rep}\,G$ & $\mathrm{Rep}\,H$.

① The first one should make you think of $\mathrm{Rep}\,S_n$, with

$V_\lambda \mapsto V_{\lambda^t} \cong V_\lambda \otimes V^{sgn}$, where λ^t is the conjugate partition of λ.

Concretely, the natural projection $\pi : G \to G/H \cong S_2$ induces an injective map $\pi^* : \mathrm{Irrep}(S_2) \to \mathrm{Irrep}(G)$.

- $\pi^*(V_{S_2}^{tr}) = V_G^{tr}$
- $\pi^*(V_{S_2}^{sgn}) =: V_G^{sgn}$

Now, for $V \in \mathrm{Rep}(G)$, $V' := V \otimes V_G^{sgn}$. That is, $V' = (V, \underset{sgn}{\circ})$,

with $g \underset{sgn}{\circ} v = \begin{cases} g\cdot v & \text{if } g \in H, \\ -g\cdot v & \text{if } g \notin H. \end{cases}$

Properties:

- $(V')' \cong V$ as G-reps
- $\chi^{V'}(g) = \begin{cases} \chi^V(g), & g \in H \\ -\chi^V(g), & g \notin H \end{cases}$
- $\mathrm{Res}^G_H(V') \cong \mathrm{Res}^G_H(V)$
- $V \in \mathrm{Irrep}(G) \Leftrightarrow V' \in \mathrm{Irrep}(G)$
- $(V_1 \oplus V_2)' \cong V_1' \oplus V_2'$
- $\mathrm{Ind}\,\mathrm{Res}\,V \cong V \oplus V'$

② Fix a $\underline{g_1} \in G$, $g_1 \notin H$. For $(W, \rho) \in \mathrm{Rep}(H)$, denote by W' the H-rep (W, ρ'), where $\rho'(h) = \rho(\underbrace{g_1 h g_1^{-1}}_{\in H \text{ since } H \text{ is normal}})$.

Properties:

- (W, ρ') is indeed an H-rep.
- For another choice of g_1, one gets an isomorphic H-rep.
- $(W')' \cong W$ as H-reps
- $\chi^{W'}(h) = \chi^W(g_1 h g_1^{-1})$
- $(\mathrm{Res}^G_H V)' \cong \mathrm{Res}^G_H V$
- $W \in \mathrm{Irrep}(H) \Leftrightarrow W' \in \mathrm{Irrep}(H)$.
- $(W_1 \oplus W_2)' \cong W_1' \oplus W_2'$
- $\mathrm{Res}\,\mathrm{Ind}\,W \cong W \oplus W'$

Prop. 29: $V \in \mathrm{Irrep}(G)$, $W := \mathrm{Res}^G_H V$. There are only 2 possibilities:

$\boxed{1}$ $W \in \mathrm{Irrep}(H)$, $W \cong W'$, $\quad \mathrm{Ind}\,W \cong V \oplus V'$, $V \ncong V'$.

$\boxed{2}$ $W \cong W_1 \oplus W_1'$, $W_1^{(')} \in \mathrm{Irrep}(H)$, $W_1 \ncong W_1'$, $\quad \mathrm{Ind}\,W_1 \cong \mathrm{Ind}\,W_1' \cong V \cong V'$

The W from $\boxed{1}$ & the W_1 & W_1' from $\boxed{2}$ yield a complete list of $\mathrm{Irrep}(H)$. The only repetitions are the same W in $\boxed{1}$ for V & for V'.

$\square$ $V \in \mathrm{Irrep}(G) \Rightarrow \underline{(\chi^V, \chi^V) = 1} \Rightarrow \# G = \underbrace{\sum_{h \in H}}_{= 2 \cdot \# H} \underbrace{|\chi^V(h)|^2}_{= \# H \cdot (\chi^W, \chi^W)_H} + \underbrace{\sum_{g \in G \setminus H} |\chi^V(g)|^2}_{\geq 0}$.

So, $\underline{(\chi^W, \chi^W)_H \in \{1, 2\}}$.

$\boxed{1}$. $(\chi^W, \chi^W)_H = 1 \Rightarrow W \in \mathrm{Irrep}(H)$.

$\Downarrow$

- $\exists g \in G \setminus H$ s.t. $\chi^V(g) \neq 0 \Rightarrow V \not\cong V'$.
- $(\chi^V, \chi^{\mathrm{Ind}\, W}) \overset{\text{Frobenius}}{=} (\chi^{\mathrm{Res}\, V}, \chi^W) = (\chi^W, \chi^W) = 1$

  $(\chi^{V'}, \chi^{\mathrm{Ind}\, W}) \overset{\text{Fr.}}{=} (\chi^{\mathrm{Res}\, V'}, \chi^W) \underset{\substack{\Uparrow \\ \mathrm{Res}\, V = \mathrm{Res}\, V'}}{=} (\chi^{\mathrm{Res}\, V}, \chi^W) = 1$

  So V & V' have multiplicity 1 in W.

  But $\underline{\dim \mathrm{Ind}\, W = 2 \dim W = 2 \dim V = \dim V + \dim V'}$.

  Conclusion: $\mathrm{Ind}\, W \cong V \oplus V'$.

- $\underline{(\mathrm{Res}\, V)' \cong \mathrm{Res}\, V}$ is a general property. Indeed, $\chi^{W'}(h) = \chi^W(g_1 h g_1^{-1}) =$

  (here: $W' \cong W$)

  $= \chi^V(g_1 h g_1^{-1}) \underset{\substack{\Uparrow \\ \chi^V \text{ is a central function.}}}{=} \chi^V(h) = \chi^W(h)$

$\boxed{2}$. $(\chi^W, \chi^W)_H = 2 \Rightarrow W \cong W_1 \oplus W_2$, $W_i \in \mathrm{Irrep}(H)$, $W_1 \not\cong W_2$.

$\Downarrow$

- $\forall g \in G \setminus H$, $\chi^V(g) = 0 \Rightarrow \chi^V = \chi^{V'} \Rightarrow V \cong V'$.
- $(\chi^V, \chi^{\mathrm{Ind}\, W_i}) = 1 \Rightarrow \mathrm{Ind}\, W_i = V \oplus \text{smth} \Rightarrow \dim V \leq \dim \mathrm{Ind}\, W_i = 2 \dim W_i$,

  But $\dim V = \dim W = \dim W_1 + \dim W_2 \Rightarrow \dim V = \dim \mathrm{Ind}\, W_i$

  $\Rightarrow \mathrm{Ind}\, W_i \cong V$, $i \in \{1, 2\}$.

- $W \cong W' \Rightarrow$ either $W_2 \cong W_1'$,

  or $W_i \cong W_i'$, $i \in \{1, 2\}$.

  The 2nd case is impossible: it would imply $W_i \oplus W_i \cong W_i \oplus W_i' \cong \mathrm{Res}\, \mathrm{Ind}\, W_i \cong \mathrm{Res}\, V = W \cong W_1 \oplus W_2 \Rightarrow W_1 \cong W_2$.

The list is complete since $\forall U \in \mathrm{Irrep}$, $\underline{\mathrm{Res}\, \mathrm{Ind}(U)}$ has a $\underline{\text{direct summand } \cong U}$.

Finally, $\underline{\mathrm{Ind} \text{ allows one to recover } \{V, V'\} \text{ from } W \text{ or } W_1}$ $\boxtimes$

character tables of G & H almost determine each other, except for the splittings $\chi^W = \chi^{W_1} + \chi^{W'}$ & $\chi^{\mathrm{Ind}\,W} = \chi^V + \chi^{V'}$.

A typical problem:

	$[h_1]$	$[h_2]$	$[g_1 h_2 g_1^{-1}]$
W_1	$a/2$	x	$b-x$
W_1'	$a/2$	$b-x$	x
$W_1 \oplus W_1' \cong W$	a	b	b

here $h_1 \sim g_1 h_1 g_1^{-1}$, but $h_2 \not\sim g_1 h_2 g_1^{-1}$ $\Big\}$ conjugacy in H

How to determine x?

Possible approach: orthogonality relations.

We are now ready for our main example:

$\boxed{G = S_n,\ H = A_n,\ g_1 = (12)}$ For $\sigma \in A_n$, put $\sigma' = (12)\sigma(12)$.

For $V \in \mathrm{Rep}(S_n)$, $V' \cong V \otimes V^{sgn}$, so $V_\lambda' \cong V_{\lambda^t}$.

The "split" situation $\boxed{2}$ occurs iff $\lambda = \lambda^t$.

Further, $\sigma \not\sim \sigma'$ in A_n $\Leftrightarrow$ σ of cycle type $(\mu_1, \dots, \mu_k)$, where $\mu_1 > \dots > \mu_k$ are all odd.

Indeed, • $\exists$ even μ_s, corresponding to a cycle τ in σ

$\Rightarrow \tau\sigma\tau^{-1} = \sigma \Rightarrow \sigma' = (12)\sigma(12) = \underbrace{(12)\tau}_{\in A_n}\sigma\tau^{-1}(12) \Rightarrow \sigma' \sim \sigma$ in A_n.

• $\forall \mu_s$ odd &

$\exists \mu_s = \mu_{s+1}$, corresponding to cycles $(a_1 \dots a_{\mu_s})$ & $(b_1 \dots b_{\mu_s})$

$\Rightarrow \tau\sigma\tau^{-1} = \sigma$ for $\tau = (a_1 b_1) \dots (a_{\mu_s} b_{\mu_s})$, which is odd

$\Rightarrow \sigma' = \underbrace{(12)\tau}_{\in A_n}\sigma\tau^{-1}(12) \Rightarrow \sigma' \sim \sigma$ in A_n

• $\forall \mu_s$ odd & $\mu_s > \mu_{s+1}$ & $\tau\sigma\tau^{-1} = \sigma$, $\sigma = \theta_1 \dots \theta_k$ (cycle decomposit^n)

$\Rightarrow \tau = \theta_1^{d_1} \dots \theta_k^{d_k}$ for some $d_i \in \mathbb{N} \cup \{0\} \Rightarrow \tau$ even,

so $(12)\sigma(12) \neq \xi\sigma\xi^{-1}$ for $\xi \in A_n$ (i.e., even).

Lemma 30: $\{\lambda \vdash n \mid \lambda = \lambda^t\} \xleftrightarrow{1:1} \{\mu \vdash n \mid \forall \mu_s \text{ is odd } \& \mu_1 > \mu_2 > \dots\}$

$\lambda \overset{H}{\longmapsto} (h(c_{11}), h(c_{22}), \dots)$

↑ hook length

Prop. 3.1: Take $\lambda \vdash n$, $\lambda = \lambda'$, $\sigma \in A_n$ of cycle type μ.

Put $\chi^+ = \chi^{W_1}$, $\chi^- = \chi^{W_1'}$, $\chi = \chi^{V_\lambda}$,

where $W = \operatorname{Res} V_\lambda \cong W_1 \oplus W_1'$

- if $\mu \neq H(\lambda)$, then $\chi^\pm(\sigma) = \chi^\pm(\sigma') = \frac{1}{2}\chi(\sigma)$.
- if $\mu = H(\lambda)$, then $\chi^\pm(\sigma) = \chi^\mp(\sigma') = \frac{1}{2}\left((-1)^m \pm \sqrt{(-1)^m \mu_1 \dots \mu_k}\right)$,

$m = \frac{1}{2}(n-k)$.

Ex.: $n=4$, $\lambda = (2,2) = \lambda'$ $\qquad H(\lambda) = (3,1)$

$m = \frac{1}{2}(4-2) = 1$, $\frac{1}{2}\left((-1)^m \pm \sqrt{(-1)^m \mu_1 \dots \mu_k}\right) = \frac{1}{2}\left(-1 \pm \sqrt{-3}\right) = e^{\pm\frac{2\pi i}{3}}$.

Recall the character table for S_4:

S_4	Id	(12) not in A_4	(123) split	(1234) not in A_4	(12)(34)
V^{tr}	1	1	1	1	1
$(V^{tr})' \cong V^{sgn}$	1	-1	1	-1	1
V^{st}	3	1	0	-1	-1
$(V^{st})'$	3	-1	0	1	-1
split $\to$ W	2	0	-1	0	2

	A_4	Id	(123)	(213)	(12)(34)
	$\operatorname{Res} V^{tr}_{S_4} \cong V^{tr}_{A_4}$	1	1	1	1
	$\operatorname{Res} V^{st}$	3	0	0	-1
$\operatorname{Res} W \cong W_1 \oplus W_1'$	W_1	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	1
	W_1'	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	1

Exo: Character table for A_5.