2. Representations of S_3 Seen Geometrically and Algebraically

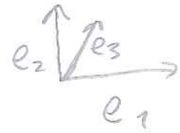
Today we look at reps of the symmetric group S_3 over the fields $k = \mathbb{R}$ & $k = \mathbb{C}$.

Recall the permutation rep. $V^{perm} = \underbrace{ke_1 + ke_2 + ke_3}_{\substack{\text{vector space over } k,\\ \text{with a basis } (e_1, e_2, e_3).}}$,

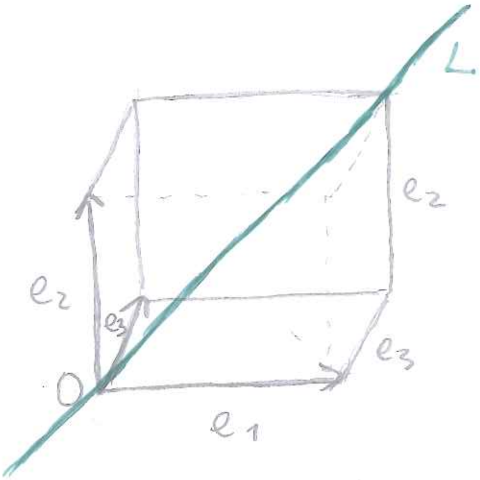
with $\sigma \cdot e_i = e_{\sigma(i)}$, $\sigma \in S_3$.

Ⓐ Geometrically ($k = \mathbb{R}$):

V^{perm} is the plane $\mathbb{R}^3$ with a chosen basis:



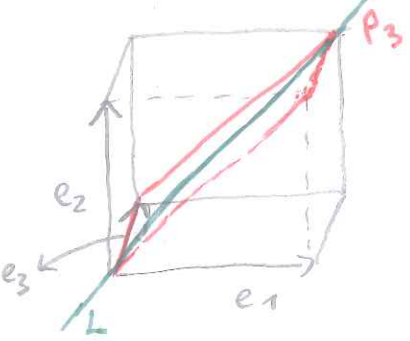
$(1\,2\,3) \in S_3$ acts on $\mathbb{R}^3$ by a rotation around the axis L with the angle $\frac{2\pi}{3}$:



$v_L = e_1 + e_2 + e_3$ is a direction vector of L

$(1\,3\,2)$ acts by a rotation around L with angle $-\frac{2\pi}{3}$

(1 2) acts by reflexion w.r.t. the plane P_3:



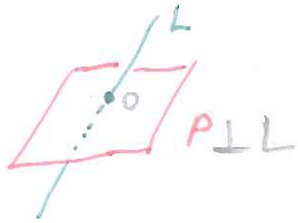
$P_3 \supset L$

(1 3) & (2 3) act by reflexions w.r.t. similar planes P_2 & P_1, both of which contain L.

$Id \in S_3$ acts by identity.

We see that all $\sigma \in S_3$ fix L point-wise.

Moreover, they fix the plane P (as a whole, not point-wise this time). Here P is the plane passing through 0 orthogonally to L.



The reason: rotations & reflexions preserve angles, so if σ fixes L, it sends things orthogonal to L to other things orthogonal to L.

Thus the S_3-action preserves the vector space decomposition $V^{perm} = L \oplus P$.

$$\begin{array}{cccc} \dim = & 3 & 1 & 2 \end{array}$$

P does not decompose further, since $(1\,2\,3)$ rotates any line in P by $\frac{2\pi}{3}$, and thus doesn't fix it.

Let us identify L & P with the reps we have already seen:

- $\underline{L \cong V^{tr}}$ (the trivial rep. of degree 1).

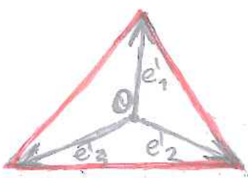
- $P = \{ v \in \mathbb{R}^3 \mid v \perp v_L \}$

$$= \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \;\middle|\; \left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = 0 \right\}$$

$$= \left\{ \sum_{i=1}^{3} \alpha_i e_i \;\middle|\; \sum_{i=1}^{3} \alpha_i = 0,\ \alpha_i \in \mathbb{R} \right\}$$

A basis of P: $(e_1 - e_3\,,\ e_2 - e_3)$.

P contains the vectors $e'_i := e_1 + e_2 + e_3 - 3e_i$, $i = 1, 2, 3$. The S_3-action permutes the e'_i: $\sigma . e'_i = e'_{\sigma(i)}$.
In particular, the $\frac{2\pi}{3}$-rotation realised by $(1\,2\,3)$ acts by $e'_1 \to e'_2 \to e'_3 \to e'_1$. So the triangle

$T =$  $\subset P$

is equilateral. Our S_3-action on $\underline{P}$ is then precisely the action of S_3 on $\mathbb{R}^2$ fixing the triangle T, seen in Lecture 1.

P is called the standard rep. of S_3, denoted by V^{st}.

This is the general pattern we'll prove for reps of finite groups: they decompose into elementary pieces (indecomposable reps), belonging to some finite list.

⚠

The geometric approach is nice for getting intuitions, but difficult to implement when dimensions increase. We'll thus describe the same reps algebraically.

Ⓑ Algebraically ($k=\mathbb{R}$ or $k=\mathbb{C}$)

(or any field of characteristic 0)

$L \overset{\text{is defined to be}}{:=} \{\alpha(e_1+e_2+e_3) \mid \alpha \in k\}$ $\qquad \dim_k L = 1$

and $V^{st} := \{\sum_{i=1}^{3} \alpha_i e_i \mid \alpha_i \in k,\ \sum_{i=1}^{3}\alpha_i = 0\}$ $\qquad \dim_k V^{st} = 2$, basis: $(e_1-e_3; e_2-e_3)$.

are subspaces of V^{perm}.

(*) $V^{perm} = L \underset{\text{direct sum}}{\oplus} V^{st}$ as vector spaces over k. Indeed,

(1) $\underline{L \cap V^{st} = \{0\}}$, since $v \in L \cap V^{st} \Rightarrow v = \alpha(e_1+e_2+e_3)$, $\alpha \in k$, and $3\alpha = 0$ (since $v \in V^{st}$) $\Rightarrow \alpha = 0 \Rightarrow v = 0$.

(2) $\underline{\underset{3}{\underset{\parallel}{\dim V^{perm}}} = \underset{1}{\underset{\parallel}{\dim L}} + \underset{2}{\underset{\parallel}{\dim V^{st}}}}$

<u>Reminder</u> (Linear algebra): (*) $\Leftrightarrow$ (1) & (2) for finite-dimensional spaces

Further, L & V^{st} are S_3-invariant:

(a) for L: $\sigma\cdot(\alpha\sum_{i=1}^{3}e_i)=\alpha\sum e_{\sigma(i)}=\alpha\sum e_i$

(b) for V^{st}: $\sigma\cdot(\sum_{i=1}^{3}\alpha_i e_i)=\sum\alpha_i e_{\sigma(i)}$, so the S_3-action preserves the sum of coefficients.

(a) also gives $L\cong V^{tr}$.

Conclusion: $V^{perm}\cong V^{tr}\oplus V^{st}$ (decomposition of S_3-reps this time)

Let us show that V^{st} is indecomposable.

Otherwise there would be an S_3-invariant subspace $W\subsetneq V^{st}$, $W\neq\{0\}$. Choose a $w=\sum\alpha_i e_i\in W$, $w\neq 0$.

One has either $\alpha_1\neq\alpha_3$ or $\alpha_2\neq\alpha_3$ (otherwise $w\in L$, and $w\in W\subset V^{st}$, but $L\cap V^{st}=\{0\}$).

Suppose, without the loss of generality, $\alpha_2\neq\alpha_3$.

$W\ni w=\alpha_1e_1+\alpha_2e_2+\alpha_3e_3$

$W\ni(23)\cdot w=\alpha_1e_1+\alpha_2e_3+\alpha_3e_2$ $\Leftarrow$ S_3-invariance

$\Rightarrow\ W\ni w-(23)\cdot w=(\alpha_2-\alpha_3)(e_2-e_3)$ $\Leftarrow$ linearity

$\Rightarrow\ W\ni e_2-e_3$ $\Leftarrow$ $\alpha_2-\alpha_3\neq 0$

$\Rightarrow\ W\ni(12)\cdot(e_2-e_3)=e_1-e_3$ $\Leftarrow$ S_3-invariance

$\Rightarrow\ W=V^{st}$, contradiction. $\Leftarrow$ (e_1-e_3,e_2-e_3) is a basis of V^{st}

We'll see the same decomposition for all groups S_n.