Lecture 19: Applications of differential calculus

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MA1S11A: Calculus with Applications for Scientists

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1 What are derivatives good for?

We have seen that derivatives are useful in a wide range of contexts. We need them to:

1) find the rate of change of a function (e.g., velocity and acceleration);
2) describe tangents to curves;
3) analyse and graph functions;
4) solve optimisation problems (i.e., maximise/minimise a quantity depending on some parameter).

Today we will comment on some of the above applications, and give two new ones:

5) Newton's method for approximating zeroes of functions;
6) L'Hôpital's rule for computing limits involving indeterminate forms of type $0/0$.

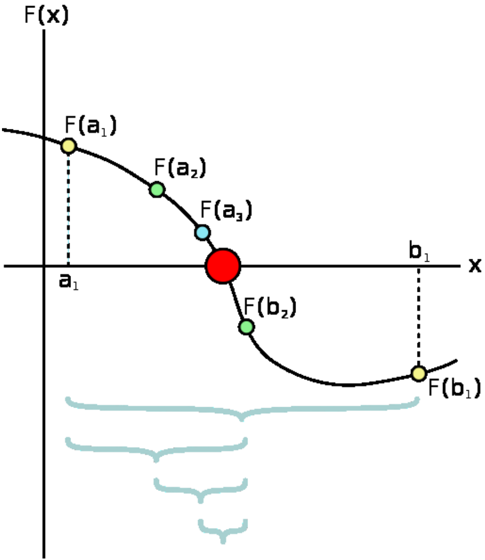
2 Newton's method

In practice, you may need to solve an equation $f(x) = 0$:

- ✓ algebraically, that is, find a real number c (called a **zero** of f) satisfying $f(c) = 0$;
- ✓ or numerically, that is, find an approximation c_0 for a zero c of f, in the sense that $|c_0 - c| < \varepsilon$, where ε is the desired precision.

For most functions, only numerical solution is feasible. That is what a computer does when asked to solve an equation. This suffices for most applications.

We have seen one method for solving equations numerically—the **bisection method**.



F(x)
F(a1)
F(a2)
F(a3)
b1
x
a1
F(b2)
F(b1)

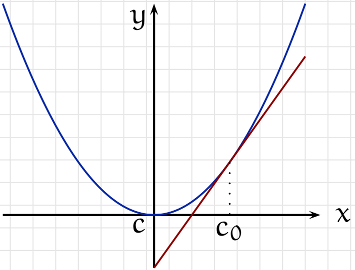
2 Newton's method

The **bisection method** applies to any continuous function f, provided that we have found some a and b with $f(a) > 0 > f(b)$.
It is based on the Intermediate Value Theorem.

This is one of the most universal methods for numerical solution of equations. However, computers prefer faster methods, like the one we are about to describe.

The **Newton's method** applies to any differentiable function f, provided that we have an initial estimate c_0 for one of its zeroes c.

We know that the best linear approximation to f at c_0 is the tangent line:



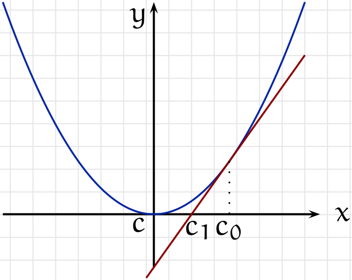
2 Newton's method

We know that the best linear approximation to f at c_0 is the tangent line

$$y = f'(c_0)(x - c_0) + f(c_0).$$

The zero of this linear function is found by solving the equation

$$0 = f'(c_0)(x - c_0) + f(c_0).$$



If $f'(c_0) \neq 0$, then the only solution is

$$c_1 = c_0 - \frac{f(c_0)}{f'(c_0)}.$$

In most cases, c_1 approximates the desired zero c better than c_0.

2 Newton's method

Iterating this procedure, we get a sequence of numbers $c_0, c_1, c_2, \dots$, defined recursively by

$$c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)}.$$

In many cases, these numbers approach the target zero c of f very fast.

Good example. The equation $x^3 - x - 1 = 0$ can be solved algebraically, but the formulas involved are too complicated.

Let us solve the equation $x^3 - x - 1 = 0$ numerically. Since $f'(x) = 3x^2 - 1$, Newton's method suggests

$$c_{n+1} = c_n - \frac{c_n^3 - c_n - 1}{3c_n^2 - 1} = \frac{2c_n^3 + 1}{3c_n^2 - 1}.$$

Starting from the initial approximation $c_0 = 1.5$, we get

$$c_1 \approx 1.34782609,$$
$$c_2 \approx 1.32520040,$$
$$c_3 \approx 1.32471817,$$
$$c_4 \approx 1.32471796,$$

2 Newton's method

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$$\begin{aligned} c_1 &\approx 1.34782609, \\ c_2 &\approx 1.32520040, \\ c_3 &\approx 1.32471817, \\ c_4 &\approx 1.32471796, \\ c_5 &\approx 1.32471796, \end{aligned}$$

after which within our precision nothing changes.

From $c_5 = c_4 - \frac{f(c_4)}{f'(c_4)}$ and $c_4 \approx c_5$, we deduce $\frac{f(c_4)}{f'(c_4)} \approx 0$. Since $f'(c_4) = 3c_4^2 - 1 \approx 4.26463302$ is far from zero, this yields $f(c_4) \approx 0$. So, c_4 is very likely to be close to a zero of f.

2 Newton's method

Bad example. Let us take $f(x) = \sqrt[3]{x}$. Newton's method yields

$$c_{n+1} = c_n - \frac{\sqrt[3]{c_n}}{\frac{1}{3\sqrt[3]{c_n^2}}} = c_n - 3c_n = -2c_n.$$

So, whichever non-zero value c_0 we start with, we get $c_n = (-2)^n c_0$, which of course gets larger and larger in magnitude, and does not converge.

2 Newton's method

Drawbacks of Newton's method:

- ✓ works for differentiable functions f (while for the bisection method a continuous f suffices);
- ✓ problems when $f'(c_n) = 0$ at some step;
- ✓ an initial approximation c_0 of a zero c is needed;
- ✓ problems for "bad" f (as in the example above) and "bad" c_0.

The initial approximation can be chosen using the bisection method.

Advantages of Newton's method:

- ✓ works very fast in most cases: e.g., for a zero c of multiplicity 1 and a good initial guess c_0, the convergence is at least quadratic (roughly, the number of correct digits doubles in every step);
- ✓ easy computation.

These advantages are sufficient for implementing the method in all computers.

Newton's method has numerous variations and generalisations.

3 L'Hôpital's rule

The derivative is defined as the limit of the difference quotient. So, we typically use limits for computing derivatives. However things can work the other way round: as some of you know from school, we sometimes use derivatives for computing limits.

Theorem (L'Hôpital's Rule). Assume that the functions f and g satisfy:

1) f and g are differentiable on (a, b), except possibly at c;
2) $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$;
3) $g'(x) \neq 0$ for all $x \in (a, c) \cup (c, b)$;
4) $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

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Example.

$$\lim_{x\to 0} \frac{\sin(x^2+x^3)}{\arcsin(x^2)} = \lim_{x\to 0} \frac{(\sin(x^2+x^3))'}{(\arcsin(x^2))'} = \lim_{x\to 0} \frac{\cos(x^2+x^3)\cdot(2x+3x^2)}{\frac{1}{\sqrt{1-x^4}}\cdot 2x}$$

$$= \frac{\cos(0)}{\frac{1}{\sqrt{1-0^4}}} \cdot \lim_{x\to 0} \frac{2+3x}{2} = \frac{1}{1}\cdot\frac{2}{2} = 1.$$

The functions $f(x) = \sin(x^2+x^3)$ and $g(x) = \arcsin(x^2)$ clearly satisfy all the conditions of L'Hôpital's rule.

3 L'Hôpital's rule

⚠ You can use L'Hôpital's rule to memorise the remarkable limits

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{(\sin(x))'}{x'}=\lim_{x\to 0}\frac{\cos(x)}{1}=\frac{\cos(0)}{1}=1,$$

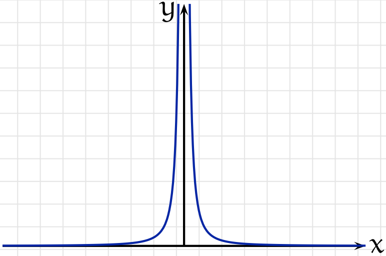
$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\lim_{x\to 0}\frac{\sin(x)}{2x}=\frac{1}{2}\lim_{x\to 0}\frac{\sin(x)}{x}=\frac{1}{2}.$$

However, this does not **prove** these limits, since we are using the derivatives $\sin' = \cos$ and $\cos' = -\sin$, whose proof exploits our remarkable limits!

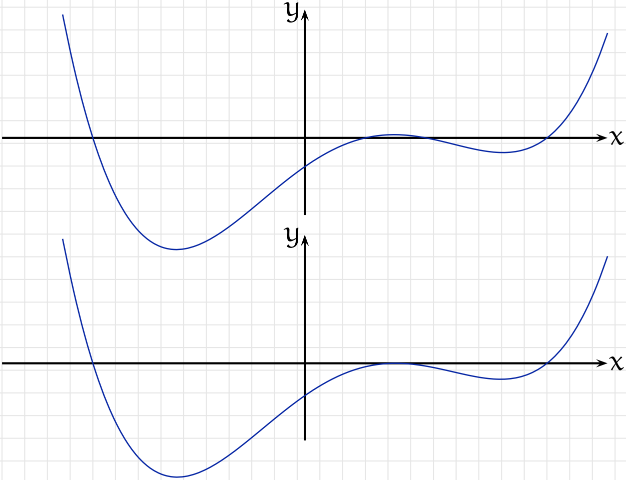
4 Derivatives and graphing

You might wonder why we care about analysing and graphing functions by hand when computers and calculators can do it faster and often better than us. An example of a powerful computer program for working with functions can be found here: `https://www.geogebra.org/graphing`

Besides understanding how such programs work, a basic knowledge of calculus allows us to interpret their results. For instance, given the graph below, you need to study the behaviour of your function at 0 to understand whether it has a vertical asymptote $x = 0$, or whether the value at 0 is finite but too large for the chosen viewing window.



Similarly, if you zoom these two graphs out, you will not distinguish them. You will need to determine, for instance, the zeroes of the two functions to feel the difference between them:



5 Derivatives and rectilinear motion

One of the original motivations for introducing derivatives was to handle velocity and acceleration. To cover these concepts in full generality, we need to work with vector-valued functions. But real-valued functions also apply in practice—to study **rectilinear motion**, i.e., motion along a line.

Consider for instance the height of an object thrown vertically, either upward or downward. From physics, you know that this motion is parabolic, and can be described by the following functions:

$$\begin{aligned} \text{height } h(t) &= at^2 + bt + c, \\ \text{velocity } h'(t) &= 2at + b, \\ \text{acceleration } h''(t) &= 2a, \end{aligned}$$

where $a, b, c \in \mathbb{R}$ are parameters. Here t is the time variable.

From physics you know that in this situation the acceleration is due to gravity only, and near the surface of the Earth its value is $\approx 9.8\,\mathrm{m}/\mathrm{s}^2$. So, you can take $a = -4.9\,\mathrm{m}/\mathrm{s}^2$. (The sign is negative since gravity acceleration is directed downward).

The motion equation becomes

$$h(t) = -4.9t^2 + bt + c.$$

We can measure the height from the point where the object was thrown, and start measuring time when this happened. This means $c = h(0) = 0$. The equation of motion now becomes

$$h(t) = -4.9t^2 + bt.$$

To determine b, you need some extra information. For example, if you are given the initial velocity v_0, then from

$$b = h'(0) = v_0.$$

Let us find stationary points of $h(t)$. Equation $h'(t) = 0$ means here $-9.8t + b = 0$, that is, $t = \frac{b}{9.8}$. If the original movement direction was upward, we get $b = v_0 > 0$, so $t = \frac{b}{9.8} > 0$ is a valid value for time. Looking at the sign of $h'(t) = -9.8t + b$, we see that in this case $h(t)$ has a local maximum at $\frac{b}{9.8}$. This corresponds to what you know from practice: a ball thrown upward stops at some moment and then falls down.

Exercise. Plot the graph of $h(t)$ for $v_0 = 5\,m/s^2$ and $v_0 = -5\,m/s^2$.

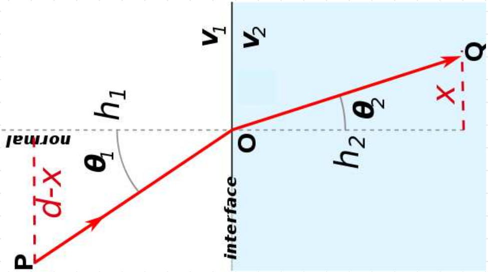
6 Derivatives and optimisation

Exercise. In optics, Fermat's principle of least time declares that a ray of light travels between two points along the path minimising its travel time.

Deduce from it Snell's law: The ratio of the sines of the angles of incidence and refraction equals the ratio of phase velocities in the two media:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

To do this, express the time traveled by the light ray in the picture below as a function f of x (here d, h_1 and h_2 are constants), and find its minimum by computing $f'(x)$. Explain why this minimum exists.



6 Derivatives and optimisation

Exercise. In optics, Fermat's principle of least time declares that a ray of light travels between two points along the path minimising its travel time.

Deduce from it the Law of reflection: In the same medium, the angle of incidence equals the angle of reflection:

$$\theta_1 = \theta_2.$$

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